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Some Elementary Inequalities Between Mean and Standard Deviation

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Abstract: Some inequalities for the mean and standard deviation of continuous probability distributions are presented here in this paper and their geometrical significance has also been discussed. It has been shown that the inequalities obtained in this paper are better than the inequalities discussed by J. Muilwijk[6].

Keywords: Random variable, Variance, Standard deviation, Infimum Supremum.

1. INTRODUCTION

If μ_1' be the mean and σ^2 the variance for continuous probability distribution function. Then variance of a random variable which varies over the interval [a, b] is bounded by the following inequality [1, 2, 3&4]:

$$0 \le \sigma^2 \le (b - \mu_1')(\mu_1' - a) \tag{1.1}$$

or

$$0 \le \sigma^2 + \left(\mu_1' - \frac{a+b}{2}\right)^2 \le \left(\frac{b-a}{2}\right)^2 \tag{1.2}$$

A bound on variance of a continuous random variable which varies over the interval [a, b] is discussed in references [3,4&6]. In this paper we try to obtain the reduction in this bound for the case when minimum and maximum values of the probability density function are prescribed. The geometrical significance of these bounds is discussed in relation with the circle diagram given in [5&7].

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2. MAIN RESULTS

Theorem 2.1

Let μ_1' be the mean and σ be the standard deviation of a continuous random variable x whose probability density function $\phi(x)$ is defined in the interval [a, b]. If m be the infimum of the function $\phi(x)$ in the interval [a, b] then we must have,

$$\frac{m(b-a)^3}{12} \leq \sigma^2 + \left(\mu_1' - \frac{a+b}{2}\right)^2 \leq \left(\frac{b-a}{2}\right)^2 - \frac{m(b-a)^3}{6}. \tag{2.1}$$

Proof

Consider

$$\sigma^2 = \int_a^b (x-\mu_1')^2 \phi(x)dx$$

$$= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \phi(x)dx - \int_a^b \left(\mu_1' - \frac{a+b}{2}\right)^2 \phi(x)dx. \tag{2.2}$$

The definite integral of probability distribution function $\phi(x)$ over the limits a to b is unity therefore from (2.2) we have

$$\sigma^2 + \left(\mu_1' - \frac{a+b}{2}\right)^2 = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \phi(x)dx. \tag{2.3}$$

$$= \left(\frac{b-a}{2}\right)^2 - \int_a^b (b-x)(x-a)\phi(x)dx. \tag{2.4}$$

Since x lies in the interval [a, b] therefore

$$(x-a)(x-b) \leq 0$$

Since m be the infimum of the function $\phi(x)$ in the interval [a, b] then Inequality (2.1) follows from (2.3) and (2.4).

Theorem 2.2

Let μ_1' be the mean and σ be the standard deviation of a continuous random variable x whose probability density function ϕ(x) is defined in the interval [a, b]. If M be the supremum of the function ϕ(x) in the interval [a, b] then we must have,

$$\left(\frac{b-a}{2}\right)^2 - \frac{M(b-a)^3}{6} \le \sigma^2 + \left(\mu_1' - \frac{a+b}{2}\right)^2 \le M\frac{(b-a)^3}{12} \tag{2.5}$$

Proof

Consider

$$\sigma^2 = \int_a^b (x-\mu_1')^2 \phi(x)dx$$

$$= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \phi(x)dx - \int_a^b \left(\mu_1' - \frac{a+b}{2}\right)^2 \phi(x)dx. \tag{2.6}$$

The definite integral of probability distribution function ϕ(x) over the limits a to b is unity therefore from (2.6) we have

$$\sigma^2 + \left(\mu_1' - \frac{a+b}{2}\right)^2 = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \phi(x)dx. \tag{2.7}$$

$$= \left(\frac{b-a}{2}\right)^2 - \int_a^b (b-x)(x-a)\phi(x)dx. \tag{2.8}$$

Since x lies in the interval [a, b] therefore

$$(x-a)(x-b) \le 0$$

Since M is the supremum of the function ϕ(x) in the interval [a, b] then Inequality (2.5) follows from (2.7) and (2.8).

3. GEOMETRICAL SIGNIFICANCE OF INEQUALITIES (2.1) AND (2.5)

i) Inequality (2.1) can be written as

$$k_1^2 \le \sigma^2 + \left(\mu_1' - \frac{a+b}{2}\right)^2 \le k_2^2, \tag{3.1}$$

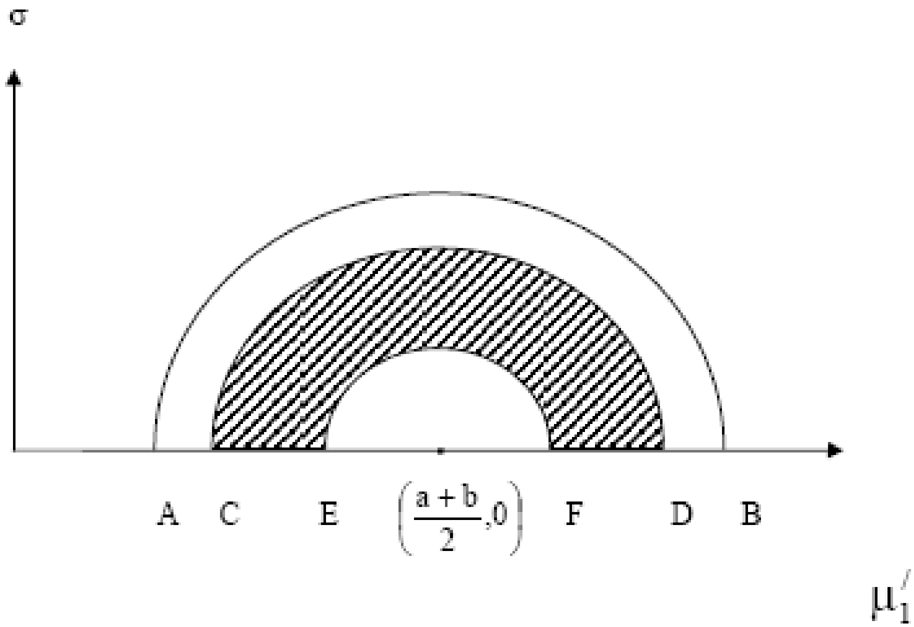

Figure: 3.1: If m be the infimum of the probability density function ϕ(x) in the interval [a, b] then from inequality (3.1) it follows that the point ($\mu_1^{/}$,σ) in $\mu_1^{/}$,σ -plane lies in a region bounded by circles with diameter EF and CD.

A ↔ (a,0), B ↔ (b,0), $E \leftrightarrow \left(\frac{a+b}{2} - k_1, 0\right)$, $F \leftrightarrow \left(\frac{a+b}{2} + k_1, 0\right)$,
$C \leftrightarrow \left(\frac{a+b}{2} - k_2, 0\right)$, and $D \leftrightarrow \left(\frac{a+b}{2} + k_2, 0\right)$

where $k_1^2 = \frac{m}{12}(b-a)^3$, and $k_2^2 = \left(\frac{b-a}{2}\right)^2 - m\frac{(b-a)^3}{6}$.

Since m(b-a) $\leq$ 1, it follows that $k_2^2 \geq k_1^2$ and $k_2^2 \leq \left(\frac{b-a}{2}\right)^2$.

From inequality (1.2) we see that the point ($\mu_1^{/}$,σ) in $\mu_1^{/}$σ- plane lies in the upper half of the circle with diameter 'b-a'. From inequality (2.1) we find that the point ($\mu_1^{/}$,σ) in $\mu_1^{/}$σ- plane lies in a region bounded by circles with radii k_1 and k_2 ($k_1 < k_2$). This is shown in **figure (3.1)**. From **figure (3.1)** we see that inequality (2.1) affects an improvement in inequality (1.2), see[6] in terms of further reducing the region in which the point ($\mu_1^{/}$,σ) must of necessity lie.

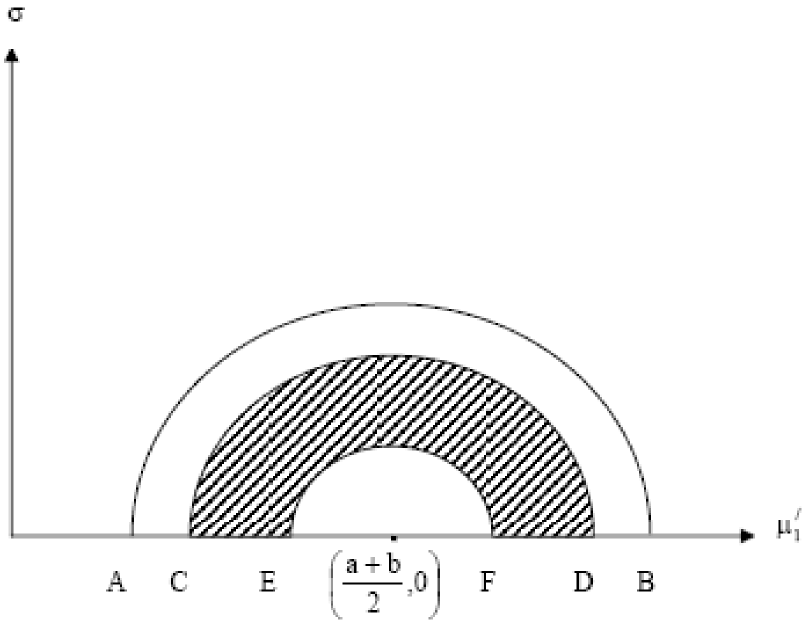

Figure: 3.2: If M be the supremum of the probability density function ϕ(x) in the interval [a, b] then from inequality (2.5) it follows that the point (μ_1',σ) in μ_1',σ -plane lies in a region bounded by circles with diameter EF and CD.

A ↔ (a, 0), B ↔ (b,0), $E \leftrightarrow \left(\frac{a+b}{2} - l_1, 0\right)$, $F \leftrightarrow \left(\frac{a+b}{2} + l_1, 0\right)$,
$C \leftrightarrow \left(\frac{a+b}{2} - l_2, 0\right)$, and $D \leftrightarrow \left(\frac{a+b}{2} + l_2, 0\right)$

ii) Inequality (2.5) can be written as

$$l_1^2 \le \sigma^2 + \left(\mu_1' - \frac{a+b}{2}\right)^2 \le l_2^2 \tag{3.2}$$

where $l_1^2 = \left(\frac{b-a}{2}\right)^2 - M\frac{(b-a)^3}{6}$, and $l_2^2 = \frac{M(b-a)^3}{12}$.

Since M(b-a) ≥ 1, it follows that $l_1^2 \le l_2^2$ and $l_2^2 \le \left(\frac{b-a}{2}\right)^2$, if and only if (b-a) M ≤ 3.
From inequality (1.2) we find that the point (μ_1',σ) in μ_1' σ- plane lies in a region bounded by circles with radii l_1 and l_2 ($l_1 < l_2$). This is shown in **figure (3.2)**. From figure we see that inequality (2.5) affects an improvement in inequality

(1.2), see[6], for (b-a) M $\leq$ 3, in terms of further reducing the region in which the point ($\mu_1^{/}$,σ) must of necessity lie.

4. CONCLUSION/RESULT ANALYSIS

A bound on variance of a continuous random variable which varies over the interval [a, b] is discussed in references [1, 3, 4&6]. Here we have obtained the reduction in this bound for the case when minimum and maximum values of the probability density function are prescribed.

REFERENCES

[1] Barnett N.S. and Drogomir S. S.. Some Elementary Inequalities for the Expectation and Variance of a Random Variable whose pdf is defined on a finite interval. RGMIA Res. Rep. Coll, **2(7)**, Art(1), (1999).

[2] Barnett N.S., Cerone P., Drogomir S. S. and Roumeliotis J.. Some Inequalities for the Expectation and Variance of a Random Variate whose pdf is n-times differentiable. JIPAM, Vol. **1(2)**, Art(21) (2000).

[3] Bhatia, R., Davis, C.. A Better Bound on the Variance. Amer. Math. Monthly, 107, 353-357(2000). http://dx.doi.org/10.2307/2589180

[4] Kapur J.N. and Anju Rani. Testing the Consistency of given values of a set of Moments of Probability Distribution. J. Bihar Math. Soc. **16**,51- 63(1995).

[5] Manju Shri Dutta, Ph.D. Thesis, On the Moment Values of Statistical Distribution, H.P. University, July, (2001).

[6] Muilwijk, J.. Note on a Theorem of M.N. Murthy and V.K. Sethi. Sankhya, Ser B, **28**, 183 (1966).

[7] Sharma S.R. and Sharma R.. On Inequalities Involving Moments of Discrete Uniform Distributions. Mathematical Journal of Interdisciplinary Sciences(MJIS),Vol. 1(1),45-56(2012). http://dx.doi.org/10.15415/mjis.2012.11003

[8] Neuman Edward, On Some Means Derived From The Schwab–Borchardt Mean II Journal of Mathematical Inequalities Volume 8, Number **2** 359–368, (2014).