DOI: 10.15415/mjis.2016.42015

A Ring-shaped Region for the Zeros of a Polynomial

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Received:July 19, 2015| Revised: December 12, 2015| Accepted: February 11, 2016

Published online: March 30, 2016

Abstract: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n .Then according to Cauchy's classical result all the zeros of P(z) lie in $|z| \le 1+A$, where $A = max_{0\le j\le n-1}\left|\frac{a_j}{a_n}\right|$. The purpose of this paper is to present a ring-shaped region containing all or a specific number of zeros of P(z).

Mathematics Subject Classification: 30C10, 30C15.

Keywords: Polynomial, Region, Zeros.

1. INTRODUCTION

The following result known as Cauchy's Theorem [2] is well known in the theory of distribution of zeros of polynomials:

Theorem A. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle $|z| \le 1+M$, where $M = max_{0\le j\le n-1}\left|\frac{a_j}{a_n}\right|$.

Several generalizations and improvements of this result are available in the literature. Mohammad [3] used Schwarz Lemma and proved the following result:

Theorem B. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in $|z| \le \frac{M}{|a_n|}$ if $|a_n| \le M$, where $M = max_{|z|=1}\left|a_n z^{n-1}+......+a_0\right| = max_{|z|=1}\left|a_0 z^{n-1}+......+a_{n-1}\right|$.

In this paper, we find a ring-shaped region containing all the zeros of P(z). We also obtain a result of similar nature for the number of zeros of the polynomial. In fact we prove the following results:

Theorem 1. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the ring-shaped region $r_1 \le |z| \le r_2$, where

$$r_1 = \frac{[R^4 |a_1|^2 (M-|a_0|)^2 + 4|a_1|^2 R^2 M^3]^{\frac{1}{2}} - |a_1| R^2 (M-|a_0|)}{2M^2},$$

$$r_2 = \frac{2{M_1}^2}{[R^4 |a_{n-1}|^2 (M_1-|a_n|)^2 + 4|a_n| R^2 M^3]^{\frac{1}{2}} - |a_{n-1}| R^2 (M_1-|a_n|)},$$

$$M = max_{|z|=R} \left| a_n z^n + ... + a_1 z \right|,$$

$$M_1 = max_{|z|=R} \left| a_0 z^n + ... + a_{n-1} z \right|,$$

R being any positive number.

Theorem 2. The number of zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n in the ring-shaped region $r_1 \le |z| \le \frac{R}{c}, 1 < c \le R$, does not exceed

$$\frac{1}{\log c} \log(1+\frac{M}{|a_0|}),$$

where M is as in Theorem 1.

2. LEMMAS

For the proofs of the above theorems we make use of the following results:

Lemma1. Let f(z) be analytic in $|z| \le 1$, f(0)=a, where $|a| < 1$, $f'(0)=b$ and $|f(z)| \le 1$ for $|z|=1$. Then, for $|z| \le 1$,

$$|f(z)| \le \frac{(1-|a|)|z|^2 + |b||z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b||z| + (1-|a|)}.$$

The example

$$f(z) = \frac{a + \frac{b}{1+a} z - z^2}{1 - \frac{b}{1+a} z - a z^2}$$

shows that the estimate is sharp.

Lemma 1 is due to Govil, Rahman and Schmeisser [1].

Lemma 2. Let f(z) be analytic for $|z| \le R$,f(0)=0, $f'(0) = b$ and $|f(z)| \le M$ for $|z| = R$.
Then, for $|z| \le R$,

$$|f(z)| \le \frac{M|z|}{R^2} \cdot \frac{M|z| + R^2|b|}{M + |b||z|}.$$

Lemma 2 is a simple deduction from Lemma 1 .

Lemma 3. If f(z) is analytic for $|z| \le R$, $f(0) \ne 0$ and $|f(z)| \le M$ for $|z| = R$, then the number of zeros of f(z)in $|z| \le \frac{R}{c}, 1 < c < R$ is less than or equal to $\frac{1}{\log c} \log \frac{M}{|f(0)|}$.

Lemma 3 is a simple deduction from Jensen's Theorem (see [4]).

Proof of Theorem 1. Let Q(z)= $a_n z^n + a_{n-1} z^{n-1} + \ldots\ldots + a_1 z$.

Then Q(z) is analytic for $|z| \le R$,Q(0)=0, $Q'(0) = a_1$ and for $|z| \le R$

$$|Q(z)| = |a_n z^n + a_{n-1} z^{n-1} + \ldots\ldots + a_1 z| \le M$$

Hence, by Lemma 2, for $|z| \le R$,

$$|Q(z)| \le \frac{M|z|}{R^2} \cdot \frac{M|z| + |a_1| R^2}{M + |a_1||z|} \quad .$$

Therefore, for $|z| \le R$,

$$\begin{aligned} |P(z)| &= |Q(z) + a_0| \\ &\ge |a_0| - |Q(z)| \\ &\ge |a_0| - \frac{M|z|}{R^2} \cdot \frac{M|z| + |a_1| R^2}{M + |a_1||z|} \\ &> 0 \end{aligned}$$

if

$$|a_0| R^2 (M + |a_1||z|) - M|z| (M|z| + |a_1| R^2) > 0$$

i.e. if $M^2 |z|^2 + |a_1| R^2 (M - |a_0|)|z| - |a_0| R^2 M < 0$

which is true if

$$|z| < \frac{[R^4 |a_1|^2 (M - |a_0|)^2 + 4|a_0| R^2 M^3]^{\frac{1}{2}} - |a_1| R^2 (M - |a_0|)}{2M^2} \quad .$$

Thus P(z) does not vanish in

$$|z| < \frac{[R^4 |a_1|^2 (M - |a_0|)^2 + 4|a_0| R^2 M^3]^{\frac{1}{2}} - |a_1| R^2 (M - |a_0|)}{2M^2} \quad .$$

On the other hand, let

$$R(z) = z^n P(\frac{1}{z}) = a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n$$
$$= H(z) + a_n \quad ,$$

where

$$H(z) = a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z \,.$$

Then H(z) is analytic and $|H(z)| \le M_1$ for $|z| \le R$, H(0)=0, $H'(0) = a_{n-1}$. Hence, by Lemma 2, for $|z| \le R$,

$$|H(z)| \le \frac{M_1 |z|}{R^2} . \frac{M_1 |z| + |a_{n-1}| R^2}{M_1 + |a_{n-1}||z|} \quad .$$

Therefore, for $|z| \le R$,

$$|R(z)| = |a_n + H(z)|$$
$$\ge |a_n| - |H(z)|$$
$$\ge |a_n| - \frac{M_1 |z|}{R^2} . \frac{M_1 |z| + |a_{n-1}| R^2}{M_1 + |a_{n-1}||z|}$$
$$> 0$$

if

$$|a_n| R^2 (M_1 + |a_{n-1}||z|) - M_1 |z| (M_1 |z| + |a_{n-1}| R^2) > 0$$

i.e. if

$$M_1^2 |z|^2 + |a_{n-1}| R^2 (M_1 - |a_n|) |z| - |a_n| R^2 M_1 < 0$$

which is true if

$$|z| < \frac{[R^4 |a_{n-1}|^2 (M_1 - |a_n|)^2 + 4|a_n| R^2 M_1{}^3]^{\frac{1}{2}} - |a_{n-1}| R^2 (M_1 - |a_n|)}{2M_1{}^2}.$$

Thus P(z) does not vanish in

$$|z| < \frac{[R^4 |a_{n-1}|^2 (M_1 - |a_n|)^2 + 4|a_n| R^2 M_1{}^3]^{\frac{1}{2}} - |a_{n-1}| R^2 (M_1 - |a_n|)}{2M_1{}^2}.$$

Hence all the zeros of R(z) lie in

$$|z| \geq \frac{[R^4 |a_{n-1}|^2 (M_1 - |a_n|)^2 + 4|a_n| R^2 M_1{}^3]^{\frac{1}{2}} - |a_{n-1}| R^2 (M_1 - |a_n|)}{2M_1{}^2}.$$

Thus all the zeros of $z^n P(\frac{1}{z})$ lie in

$$|z| \geq \frac{[R^4 |a_{n-1}|^2 (M_1 - |a_n|)^2 + 4|a_n| R^2 M_1{}^3]^{\frac{1}{2}} - |a_{n-1}| R^2 (M_1 - |a_n|)}{2M_1{}^2}.$$

Hence all the zeros of $P(\frac{1}{z})$ lie in

$$|z| \geq \frac{[R^4 |a_{n-1}|^2 (M_1 - |a_n|)^2 + 4|a_n| R^2 M_1{}^3]^{\frac{1}{2}} - |a_{n-1}| R^2 (M_1 - |a_n|)}{2M_1{}^2}.$$

This implies that all the zeros of P(z) lie in

$$|z| \leq \frac{2M_1{}^2}{[R^4 |a_{n-1}|^2 (M_1 - |a_n|)^2 + 4|a_n| R^2 M_1{}^3]^{\frac{1}{2}} - |a_{n-1}| R^2 (M_1 - |a_n|)}.$$

Consequently, it follows that all the zeros of P(z) lie in the ring-shaped region $r_1 \leq |z| \leq r_2$ and the proof of Theorem 1 is complete.

Proof of Theorem 2. To prove Theorem 2, we need to show only that the number of zeros of P(z) in $|z| \leq \frac{R}{c}, 1 < c < R$ does not exceed $\frac{1}{\log c} \log(1 + \frac{M}{|a_0|})$.

Since P(z) is analytic for $|z| \leq R$, P(0)= a_0 and for $|z| \leq R$,

$$|P(z)| = |a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0|$$

$$\leq \left|a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z\right| + |a_0|$$

$$\leq M + |a_0|$$,

it follows , by Lemma 3, that the number of zeros of P(z) in $|z| \leq \frac{R}{c}, 1 < c \leq R$ does not exceed $\frac{1}{log\, c} log \frac{M + |a_0|}{|a_0|}$ = $\frac{1}{\log c} \log(1 + \frac{M}{|a_0|})$ and Theorem 2 follows.

ACKNOWLEDGEMENT

The authors are thankful to the referee for his value-able suggestions.

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