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Characterization of Group Divisible Designs

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Abstract In this paper, some inequalities of PBIB designs has been given. In this paper, the characterization of all the three types of GD designs, by finding some equalities and inequalities among the parameters of GD designs is obtained.

Keywords: Association Scheme; Group Divisible designs.

1. INTRODUCTION

Bose and Nair (1939) introduced a class of binary, equi-replicate and proper designs, which we called PBIBD with m-associate classes. Various authors like Bose and Shimamoto (1952), Nair and Rao (1942), discussed various properties, method of construction and applications of PBIB designs. Bose and Shimamoto (1952) classified the PBIBD design of two-associate classes each of size k (k < v).

Nair and Rao (1942) modified the original definition of PBIB designs. Further, Bose and Shimamoto (1952) classified the known PBIB designs with two-associate classes into following:

(i) Group Divisible (GD), (ii) Simple (S.I), (iii) Triangular (T), (iv) Latin Square Type (L_i), (v) Cyclic Designs.

Clatworthy (1973), extensively tabulated GD designs. Since then Freeman (1976), Kageyama (1985a, 1985b), Bhagwandas (1985), Dey and Nigam (1985), have reported several methods of constructing GD designs.

A group divisible design is an arrangement of v(=mn) treatments into b blocks such that each block contains k(<v) distinct treatments which are partitioned into m($\geq$ 2) groups of n($\geq$2) treatments each, further any two distinct treatments occurring together in λ_1 blocks if they belong to the same group, and in λ_2 blocks if they belong to different groups. Bose and Connor

(1952) further subdivided the group divisible designs into following three classes depending on the values of the characteristic roots.

(1) Singular Group Divisible Design (SGD)
(2) Semi-Regular Group Divisible Design (SRGD) and
(3) Regular Group Divisible Design (RGD)

A Group Divisible Design is said to be Singular Group Divisible Design if (r-λ1) = 0, a Group Divisible Design is said to be Semi-Regular Group Divisible Design if $(r-\lambda_1)>0$ and $(rk-v\lambda_2) = 0$, and a Group Divisible Design is said to be Regular Group Divisible Design if $(r-\lambda_1) > 0$ and $(rk\text{-}v\lambda_2)>0$.

Ghosh and Das (1993) developed the two way group divisible designs with partial balance for group comparisons. For this study examples were also carried out. Ghosh and Divecha (1995) constructed four new semi-regular GD designs which are obtained by using orthogonal main effect plan of size 50 in 11 factors.

Here, the characterization of all the three types of GD designs, by finding some equalities and inequalities among the parameters of GD designs, which is discussed in section (2) is obtained. A list of GD designs, which satisfied this inequality, is shown in Appendix A.

2. CHARACTERIZATION

Let v, b, r, k, λ_1, λ_2, m, n are the parameters of SRGD design. Here we have made some efforts to obtain inequality criteria of a SRGD design among the parameters v, b, r, k and p, where $p = \lambda_2$. This is shown in Theorems 2.1 to 2.6. Here we are considering only those SRGD designs for which m ≠ k.

2.1 Inequalities regarding SRGD Design

Theorem 2.1: For any SRGD designs $\frac{v}{k} = \frac{r}{p}$ holds true, if $\lambda_1 = 0$ and $\lambda_2 = p$, where *p=1, 2, 3, 4, 5.*

Proof: Let the parameters of SRGD design are v, b, r, k, λ_1, λ_2, m, n. We know that the following parametric relations always holds true for any PBIB deisgn

$$rv = bk \text{ and} \sum n_i\lambda_i = r(k-1) \tag{1}$$

Again for a GD design, we have

$$n = n_1 + 1, \text{ and } m = \frac{n_2}{n} + 1 \tag{2}$$

From eq. (1) we have

$$n_1\lambda_1 + n_2\lambda_2 = r(k-1)$$

Using $\lambda_1 = 0$ and $\lambda_2 = p$, we get

$$n_2 p = r(k-1) \tag{3}$$

$$\text{As for any PBIB design } \frac{v}{k} = \frac{b}{r} = n \tag{4}$$

So eq. (4) is true for any GD design.
Taking m = k and using eq. (2), we have

$$\frac{n_2}{n} + 1 = m \quad \text{That is,} \quad n_2 = n(k-1) \tag{5}$$

Using eq. (3) and eq. (5), we have

$$n = \frac{r}{p} \tag{6}$$

Using eq. (4) and eq. (6), we have

$$vp = rk \tag{7}$$

Which also prove that, $rk - v\lambda_2 = 0$ holds true for a SRGD design (as $\lambda_2 = p$)

Remarks 2.1: $rk - v\lambda_2 = 0$ hold true for any SRGD and is available in the literature however it has been proved using the characteristic roots of NN' matrix of a SRGD designs However we proved it using simple technique.

(1) In a SRGD design, rk is always divisible by λ_2, which can easily be seen from eq. (7).
(2) From eq. (7) we have, $vp = rk$,
(3) If r=k then $vp = r^2$ and hence $r = \sqrt{v\lambda_2}$

This shows that if $r = k$, $v\lambda_2$ must be a perfect square (from eq. 7) for SRGD designs

(4) If $\lambda_1 = 0$, $\lambda_2 = p$, then using eq. (3), we can express p_{22}^1 in terms of r, k and λ_2 which is given below:

$$p_{22}^{1} p = r(k-1) \qquad \text{(As } n_2 = p_{22}^{1} = n(m-1)\text{)}$$

$$p_{22}^{1} = \frac{r(k-1)}{\lambda_2}$$

Where $\lambda_2 = p$ Since p_{22}^{1} is an integer and hence it shows that $r(k-1)$ must be divisible by λ_2 and both $r(k-1)$ as λ_2 are integers.

Example 2.1: Consider SR-1 (Clatworthy (1973)) with parameters *v=b=4, r=k=2, m=2, n=2,* $\lambda_1=0$, $\lambda_2 = 1$. This satisfies the theorem 2.1. Similarly, SR-18 whose parameters are *v=6, b=4, r=2, k=3, m=3, n=2,* $\lambda_1=0$, $\lambda_2 = 1$ also satisfies the Theorem 2.1.

Theorem 2.2: $b \geq v + r - k$ holds true for SRGD and RGD design, provided $\lambda_1 = 0, \lambda_2 = 1$ and $r \geq k$.

Proof: Let $b \geq v + r - k$ holds true,

That is $b - v - r + k \geq 0$

$$\frac{rv}{k} - v - r + k \geq 0 \qquad \text{(Since } rv = bk\text{)} \qquad (8)$$

But for SRGD design we know that, $v = rk$, as $\lambda_2 = 1$

So from eq. (8), We have $r^2 - rk - r + k \geq 0 = (r-k)(r-1) \geq 0$

Therefore, $r \geq k$, which is true as $r \geq 1$ holds true always.

Hence $b \geq v + r - k$, for any SRGD design, when $\lambda_1 = 0$, $\lambda_2 = 1$ and $r \geq k$.

Remark 2.2: (**1**) Bose (1939) has shows that $b \geq v + r - k$ holds true for a BIB designs, However it is also true for SRGD design when $\lambda_1 = 0, \lambda_2 = 1$ *and* $r \geq k$.

(**2**) SRGD for which $b \geq v + r - k$ are resolvable *(SR-2, SR-3, SR-4, SR-5* affine resolvable). However *SR-30, SR-53, SR- 55, SR-67, SR-70, SR-83, SR-93, SR-101, SR-108* satisfy $b \geq v + r - k$, as $r \geq k$ but are non-resolvable.

(**3**) when $b \geq v + r - k$ is satisfied then λ_1 is zero.

Example 2.2: Consider SR-26 (Clatworthy 1973) with parameters *v=12, b=16, r=4, k=3, n=3, n=4,* $\lambda_1 = 0$, $\lambda_2 = 1$. This satisfies the Theorem 2.2.

Theorem 2.3: Next we generalize Theorem 2.2 for any *p* = 2, 3, 4, 5. If $r \geq k$, then $b \geq v + r - k$ holds true for which when $\lambda_1 = 0$ and $\lambda_2 = p$ (*p* = 2, 3, 4, 5).

Proof: Let $b \geq v + r - k$ holds true (Since $rv = bk$)

So, $\frac{rv}{k} - v - r + k \geq 0$ (from theorem2.1)

$$\frac{r(rk)}{k(p)} - \frac{rk}{p} - r + k \geq 0$$

$$\frac{r^2}{p} - \frac{rk}{p} - r + k \geq 0$$

$(\frac{r}{p} - 1)(r - k) \geq 0$ holds true only if

$r \geq k$, which is true (as given), so $r \geq p$, as $(p > 0)$ Which is obvious. Therefore $b \geq v + r - k$ holds true.

Example 2.3: Consider SR-2 (Clatworthy 1973) with parameters *v= 4, b=8, r=4, k=2,* $\lambda_1 = 0$, $\lambda_2 = 2$. These parameters satisfy the Theorem 2.3. Similarly SR-5 with the parameters *v=4, b=20, r= 10, k=2,* $\lambda_1 = 0$, $\lambda_2 = 5$ also satisfies the theorem 2.3.

***Theorem 2.4*:** For SRGD design $p_{12}^2\lambda_1 = n\lambda_2 - r$ holds true, when $p_{12}^2 = n - 1$

Proof: We know that $\sum n_i\lambda_i = r(k-1)$

That is, $n_1\lambda_1 + n_2\lambda_2 = r(k-1)$

Also from eq. (2) $n = n_1 + 1$ and $m = \frac{n_2}{n} + 1 \Rightarrow n_2 = n(m-1)$

$$(n-1)\lambda_1 + n\ (m-1)\lambda_2 = r(k-1) \quad (9)$$

Using eq. (7) and eq. (9), we have

$$(n-1)\lambda_1 + n\ (\frac{v}{n} - 1)\lambda_2 = r(k-1)$$

$$p_{12}^2\lambda_1 + v\lambda_2 - n\lambda_2 = rk - r$$

Finally after solving we have $p_{12}^2\lambda_1 = n\lambda_2 - r$ where $p_{12}^2 = n - 1$

Corollary 2.1: $n\lambda_2 - r = 0$ **i**f $\lambda_1 = 0$.

Theorem2.5: In case of SRGD design, if $\lambda_1 = 0$ then $\frac{b}{n} = r$

Proof: L.H.S. $\frac{b}{n} = \frac{rv}{nk}$

Again for SRGD design k=m, if $\lambda_1 = 0$

So $\frac{b}{n} = \frac{rv}{nm} = \frac{rv}{v}$

$\Rightarrow \frac{b}{n} = r$

Corollary 2.2: If $\lambda_1 \neq 0$ then $b = \frac{nr}{2}$ except *v=12, r= 10, k= 6, b= 20,* $\lambda_1 = 4$, $\lambda_2 = 3$.

Theorem 2.6: For a SRGD design if $\lambda_1 = 0$ **,** $\lambda_2 = 1$ **,** i.e. $\lambda_2 = \lambda_1 + 1$ then $v = rk$ **.**

Theorem 2.7: For a SRGD design if $r < k$ then $b \geq v - m + 1$ or $b \geq v - k + 1$.

Appendix

Table 1: List of SRGD's which satisfies the above theorems

w	r	k	b	M	N	λ_1	λ_2	P_{12}^2	v/k	r/p	$b \geq v+r-k$	$b \geq v-k+1$	b= nr	p_{12}^2 $\lambda_1 = n\lambda_2 - r$	$p_{12}^2\lambda_1$
4	2	2	4	2	2	0	1	1	2	2	4	3	4	0	0
4	4	2	8	2	2	0	2	1	2	2	6	3	8	0	0
4	6	2	12	2	2	0	3	1	2	2	8	3	12	0	0
4	8	2	16	2	2	0	4	1	2	2	10	3	16	0	0
4	10	2	20	2	2	0	5	1	2	2	12	3	20	0	0
6	3	2	9	2	3	0	1	2	3	3	7	5	9	0	0
6	6	2	18	2	3	0	2	2	3	3	10	5	18	0	0
6	9	2	27	2	3	0	3	2	3	3	13	5	27	0	0
8	4	2	16	2	4	0	1	3	4	4	10	7	16	0	0
8	8	2	32	2	4	0	2	3	4	4	14	7	32	0	0
10	5	2	25	2	5	0	1	4	5	5	13	9	25	0	0
10	10	2	50	2	5	0	2	4	5	5	18	9	50	0	0
12	6	2	36	2	6	0	1	5	6	6	16	11	36	0	0
14	7	2	49	2	7	0	1	6	7	7	19	13	49	0	0
16	8	2	64	2	8	0	1	7	8	8	22	15	64	0	0
18	9	2	81	2	9	0	1	8	9	9	25	17	81	0	0
20	10	2	100	2	10	0	1	9	10	10	28	19	100	0	0
6	2	3	4	3	2	0	1	1	2	2	5	4	4	0	0
6	4	3	8	3	2	0	2	1	2	2	7	4	8	0	0
6	6	3	12	3	2	0	3	1	2	2	9	4	12	0	0
6	8	3	16	3	2	0	4	1	2	2	11	4	16	0	0
6	10	3	20	3	2	0	5	1	2	2	13	4	20	0	0
9	3	3	9	3	3	0	1	2	3	3	9	7	9	0	0
9	6	3	18	3	3	0	2	2	3	3	12	7	18	0	0

w	r	k	b	M	N	λ_1	λ_2	P_{12}^2	v/k	r/p	b≥v+r−k	b≥v−k+1	b= nr	p_{12}^2 λ_1=nλ_2−r	$p_{12}^2\lambda_1$
9	9	3	27	3	3	0	3	2	3	3	15	7	27	0	0
12	4	3	16	3	4	0	1	3	4	4	13	10	16	0	0
12	8	3	32	3	4	0	2	3	4	4	17	10	32	0	0
15	5	3	25	3	5	0	1	4	5	5	17	13	25	0	0
15	10	3	50	3	5	0	2	4	5	5	22	13	50	0	0
18	6	3	36	3	6	0	1	5	6	6	21	16	36	0	0
21	7	3	49	3	7	0	1	6	7	7	25	19	49	0	0
24	8	3	64	3	8	0	1	7	8	8	29	22	64	0	0
27	9	3	81	3	9	0	1	8	9	9	33	25	81	0	0
30	10	3	100	3	10	0	1	9	10	10	37	28	100	0	0
6	6	4	9	2	3	3	4	2	1.5	1.5	8	3	18	6	6
8	4	4	8	4	2	0	2	1	2	2	8	5	8	0	0
8	6	4	12	4	2	0	3	1	2	2	10	5	12	0	0
8	6	4	12	2	4	2	3	3	2	2	10	5	24	6	6
8	8	4	16	4	2	0	4	1	2	2	12	5	16	0	0
8	10	4	20	4	2	0	5	1	2	2	14	5	20	0	0
12	3	4	9	4	3	0	1	2	3	3	11	9	9	0	0
12	6	4	18	4	3	0	2	2	3	3	14	9	18	0	0
12	9	4	27	4	3	0	3	2	3	3	17	9	27	0	0
16	4	4	16	4	4	0	1	3	4	4	16	13	16	0	0
16	8	4	32	4	4	0	2	3	4	4	20	13	32	0	0
20	5	4	25	4	5	0	1	4	5	5	21	17	25	0	0
20	10	4	50	4	5	0	2	4	5	5	26	17	50	0	0
28	7	4	49	4	7	0	1	6	7	7	31	25	49	0	0
32	8	4	64	4	8	0	1	7	8	8	36	29	64	0	0
36	9	4	81	4	9	0	1	8	9	9	41	33	81	0	0
40	10	4	100	4	10	0	1	9	10	10	46	37	100	0	0
10	4	5	8	5	2	0	2	1	2	2	9	6	8	0	0
10	6	5	12	5	2	0	3	1	2	2	11	6	12	0	0
10	8	5	16	5	2	0	4	1	2	2	13	6	16	0	0
10	10	5	20	5	2	0	5	1	2	2	15	6	20	0	0
15	6	5	18	5	3	0	2	2	3	3	16	11	18	0	0
15	9	5	27	5	3	0	3	2	3	3	19	11	27	0	0
20	4	5	16	5	4	0	1	3	4	4	19	16	16	0	0
20	8	5	32	5	4	0	2	3	4	4	23	16	32	0	0
25	5	5	25	5	5	0	1	4	5	5	25	21	25	0	0
25	10	5	50	5	5	0	2	4	5	5	30	21	50	0	0
35	7	5	49	5	7	0	1	6	7	7	37	31	49	0	0
40	8	5	64	5	8	0	1	7	8	8	43	36	64	0	0
45	9	5	81	5	9	0	1	8	9	9	49	41	81	0	0
9	6	6	9	3	3	3	4	2	1.5	1.5	9	4	18	6	6
12	4	6	8	6	2	0	2	1	2	2	10	7	8	0	0
12	6	6	12	6	2	0	3	1	2	2	12	7	12	0	0

w	r	k	b	M	N	λ_1	λ_2	P_{12}^2	v/k	r/p	$b \geq v+r-k$	$b \geq v-k+1$	$b = nr$	p_{12}^2 $\lambda_1 = n\lambda_2 - r$	$p_{12}^2\lambda_1$
12	6	6	12	3	4	2	3	3	2	2	12	7	24	6	6
12	8	6	16	6	2	0	4	1	2	2	14	7	16	0	0
12	10	6	20	6	2	0	5	1	2	2	16	7	20	0	0
12	10	6	20	2	6	4	5	5	2	2	16	7	60	20	20
18	6	6	18	6	3	0	2	2	3	3	18	13	18	0	0
18	9	6	27	6	3	0	3	2	3	3	21	13	27	0	0
24	8	6	32	6	4	0	2	3	4	4	26	19	32	0	0
30	5	6	25	6	5	0	1	4	5	5	29	25	25	0	0
30	10	6	50	6	5	0	2	4	5	5	34	25	50	0	0
42	7	6	49	6	7	0	1	6	7	7	43	37	49	0	0
48	8	6	64	6	8	0	1	7	8	8	50	43	64	0	0
54	9	6	81	6	9	0	1	8	9	9	57	49	81	0	0
14	4	7	8	7	2	0	2	1	2	2	11	8	8	0	0
14	6	7	12	7	2	0	3	1	2	2	13	8	12	0	0
14	8	7	16	7	2	0	4	1	2	2	15	8	16	0	0
14	10	7	20	7	2	0	5	1	2	2	17	8	20	0	0
21	6	7	18	7	3	0	2	2	3	3	20	15	18	0	0
21	9	7	27	7	3	0	3	2	3	3	23	15	27	0	0
28	8	7	32	7	4	0	2	3	4	4	29	22	32	0	0
49	7	7	49	7	7	0	1	6	7	7	49	43	49	0	0
56	8	7	64	7	8	0	1	7	8	8	57	50	64	0	0
63	9	7	81	7	9	0	1	8	9	9	65	57	81	0	0
12	6	8	9	4	3	3	4	2	1.5	1.5	10	5	18	6	6
16	6	8	12	8	2	0	3	1	2	2	14	9	12	0	0
16	8	8	16	8	2	0	4	1	2	2	16	9	16	0	0
16	10	8	20	8	2	0	5	1	2	2	18	9	20	0	0
24	9	8	27	8	3	0	3	2	3	3	25	17	27	0	0
32	8	8	32	8	4	0	2	3	4	4	32	25	32	0	0
56	7	8	49	8	7	0	1	6	7	7	55	49	49	0	0
64	8	8	64	8	8	0	1	7	8	8	64	57	64	0	0
72	9	8	81	8	9	0	1	8	9	9	73	65	81	0	0
18	6	9	12	9	2	0	3	1	2	2	15	10	12	0	0
18	8	9	16	9	2	0	4	1	2	2	17	10	16	0	0
18	10	9	20	9	2	0	5	1	2	2	19	10	20	0	0
27	9	9	27	9	3	0	3	2	3	3	27	19	27	0	0
36	8	9	32	9	4	0	2	3	4	4	35	28	32	0	0
72	8	9	64	9	8	0	1	7	8	8	71	64	64	0	0
81	9	9	81	9	9	0	1	8	9	9	81	73	81	0	0
20	6	10	12	10	2	0	3	1	2	2	16	11	12	0	0
20	8	10	16	10	2	0	4	1	2	2	18	11	16	0	0
20	10	10	20	10	2	0	5	1	2	2	20	11	20	0	0
30	9	10	27	10	3	0	3	2	3	3	29	21	27	0	0
90	9	10	81	10	9	0	1	8	9	9	89	81	81	0	0

Table 2: List of RGD's which satisfy the above theorems.

v	r	K	b	M	n	λ_1	Λ_2	b ≥v+r-k
4	4	2	8	2	2	2	1	6
4	5	2	10	2	2	3	1	7
4	5	2	10	2	2	1	2	7
4	6	2	12	2	2	4	1	8
4	7	2	14	2	2	5	1	9
4	7	2	14	2	2	3	2	9
4	7	2	14	2	2	1	3	9
4	8	2	16	2	2	6	1	10
4	8	2	16	2	2	4	2	10
4	8	2	16	2	2	2	3	10
4	9	2	18	2	2	7	1	11
4	9	2	18	2	2	5	2	11
4	9	2	18	2	2	1	4	11
4	10	2	20	2	2	8	1	12
4	10	2	20	2	2	6	2	12
4	10	2	20	2	2	4	3	12
4	10	2	20	2	2	2	4	12
6	4	2	12	3	2	0	1	8
6	6	2	18	3	2	3	1	10
6	7	2	21	2	3	2	1	11
6	7	2	21	3	2	3	1	11
6	8	2	24	3	2	4	1	12
6	8	2	24	3	2	0	2	12
6	8	2	24	2	3	1	2	12
6	9	2	27	2	3	3	1	13
6	9	2	27	3	2	5	1	13
6	9	2	27	3	2	1	2	13
6	10	2	30	3	2	6	1	14
8	6	2	24	4	2	0	1	12
8	8	2	32	4	2	2	1	14
8	9	2	36	4	2	3	1	15
8	10	2	40	2	4	2	1	16
8	10	2	40	4	2	4	1	16
9	6	2	27	3	3	0	1	13
9	10	2	45	3	3	2	1	17

v	r	K	b	M	n	λ_1	Λ_2	b ≥v+r-k
10	8	2	40	5	2	0	1	16
10	10	2	50	5	2	2	1	18
12	8	2	48	3	4	0	1	18
12	9	2	54	4	3	0	1	19
12	10	2	60	6	2	0	1	20
15	10	2	75	3	5	0	1	23
6	3	3	6	3	2	2	1	6
6	6	3	12	2	3	3	2	9
6	6	3	12	3	2	4	2	9
6	7	3	14	2	3	4	2	10
6	7	3	14	3	2	2	3	10
6	8	3	16	2	3	5	2	11
6	8	3	16	3	2	4	3	11
6	9	3	18	2	3	6	2	12
6	9	3	18	3	2	6	3	12
6	9	3	18	3	2	2	4	12
6	9	3	18	2	3	3	4	12
6	10	3	20	2	3	7	2	13
8	3	3	8	4	2	0	1	8
8	6	3	16	4	2	0	2	11
8	9	3	24	4	2	6	2	14
8	9	3	24	4	2	0	3	14
8	9	3	24	2	4	2	3	14
9	5	3	15	3	3	2	1	11
9	6	3	18	3	3	3	1	12
9	7	3	21	3	3	4	1	13
9	7	3	21	3	3	1	2	13
9	8	3	24	3	3	5	1	14
9	9	3	27	3	3	6	1	15
9	9	3	27	3	3	3	2	15
9	10	3	30	3	3	7	1	16
9	10	3	30	3	3	4	2	16
9	10	3	30	3	3	1	3	16
10	6	3	20	5	2	4	1	13
12	5	3	20	6	2	0	1	14
12	6	3	24	6	2	2	1	15

v	r	K	b	M	n	λ_1	Λ_2	b ≥v+r-k
12	7	3	28	6	2	4	1	16
12	8	3	32	2	6	2	1	17
12	9	3	36	4	3	0	2	18
12	10	3	40	3	4	4	1	19
12	10	3	40	6	2	0	2	19
12	10	3	40	4	3	1	2	19
14	6	3	28	7	2	0	1	17
14	9	3	42	7	2	6	1	20
15	6	3	30	5	3	0	1	18
15	8	3	40	5	3	2	1	20
15	9	3	45	3	5	2	1	21
15	9	3	45	5	3	3	1	21
15	10	3	50	5	3	4	1	22
16	6	3	32	4	4	0	1	19
16	9	3	48	4	4	2	1	22
18	8	3	48	9	2	0	1	23
18	9	3	54	9	2	2	1	24
20	9	3	60	10	2	0	1	26
21	9	3	63	7	3	0	1	27
24	9	3	72	4	6	0	1	30
24	10	3	80	6	4	0	1	31
6	4	4	6	2	3	3	2	6
6	8	4	12	2	3	6	4	10
6	8	4	12	3	2	4	5	10
8	5	4	10	4	2	3	2	9
8	8	4	16	2	4	4	3	12
8	8	4	16	4	2	6	3	12
8	9	4	18	2	4	5	3	13
8	9	4	18	4	2	3	4	13
8	10	4	20	2	4	6	3	14
8	10	4	20	4	2	6	4	14
9	4	4	9	3	3	3	1	9
9	8	4	18	3	3	6	2	13
10	8	4	20	5	2	0	3	14
10	10	4	25	2	5	5	2	16
10	10	4	25	5	2	6	3	16

v	r	K	b	M	n	λ_1	Λ_2	b ≥v+r-k
12	4	4	12	6	2	2	1	12
12	8	4	24	6	2	4	2	16
12	10	4	30	3	4	2	3	18
14	4	4	14	7	2	0	1	14
14	8	4	28	7	2	0	2	18
15	4	4	15	5	3	0	1	15
15	8	4	30	5	3	6	1	19
15	8	4	30	5	3	0	2	19
15	8	4	30	3	5	1	2	19
16	6	4	24	4	4	2	1	18
16	7	4	28	4	4	3	1	19
16	8	4	32	4	4	4	1	20
16	9	4	36	4	4	5	1	21
16	9	4	36	6	2	2	1	21
16	10	4	40	4	4	6	1	22
20	9	4	45	4	5	3	1	25
24	7	4	42	8	3	0	1	27
24	9	4	54	8	3	3	1	29
24	10	4	60	3	8	2	1	30
26	8	4	52	13	2	0	1	30
27	8	4	54	9	3	0	1	31
28	8	4	56	7	4	0	1	32
28	10	4	70	7	4	2	1	34
30	10	4	75	15	2	2	1	36
8	5	5	8	2	4	4	2	8
8	5	5	8	4	2	2	3	8
8	10	5	16	2	4	8	4	13
8	10	5	16	4	2	4	6	13
9	5	5	9	3	3	4	2	9
9	10	5	18	3	3	8	4	14
10	5	5	10	5	2	4	2	10
10	7	5	14	5	2	4	3	12
10	10	5	20	2	5	5	4	15
10	10	5	20	5	2	8	4	15
12	5	5	12	3	4	4	1	12
12	5	5	12	6	2	0	2	12

v	r	K	b	M	n	λ_1	Λ_2	b ≥v+r-k
12	5	5	12	4	3	1	2	12
12	10	5	24	3	4	8	2	17
12	10	5	24	6	2	0	4	17
12	10	5	24	4	3	2	4	17
15	10	5	30	5	2	8	2	20
15	10	5	30	5	3	2	3	20
18	10	5	36	9	2	8	2	23
20	10	5	40	5	4	8	1	25
24	5	5	24	6	4	0	1	24
24	10	5	48	6	4	0	2	29
25	7	5	35	5	5	2	1	27
25	8	5	40	5	5	3	1	28
25	9	5	45	5	5	4	1	29
25	10	5	50	5	5	5	1	30
35	10	5	70	5	7	2	1	40
39	10	5	78	13	3	2	1	44
40	9	5	72	10	4	0	1	44
44	10	5	88	11	4	0	1	49
45	10	5	90	9	5	0	1	50
8	9	6	12	2	4	7	6	11
9	10	6	15	3	3	7	6	13
10	6	6	10	2	5	5	2	10
12	9	6	18	3	4	7	3	15
15	6	6	15	3	5	5	1	15
18	8	6	24	6	3	5	2	20
27	6	6	27	9	3	3	1	27
28	6	6	28	7	4	2	1	28
9	7	7	9	3	3	6	5	9
12	7	7	12	2	6	6	2	12
12	7	7	12	3	4	6	3	12
12	7	7	12	6	2	2	4	12
12	7	7	12	4	3	3	4	12
14	7	7	14	7	2	6	3	14
18	7	7	18	3	6	6	1	18
20	7	7	20	4	5	3	2	20
20	7	7	20	10	2	6	2	20

v	r	K	b	M	n	λ_1	Λ_2	b ≥v+r-k
28	10	7	40	4	7	3	2	31
33	7	7	33	3	11	2	1	33
48	7	7	48	8	6	0	1	48
49	9	7	63	7	7	2	1	51
49	10	7	70	7	7	3	1	52
12	8	8	12	6	2	6	5	12
14	8	8	14	2	7	7	2	14
21	8	8	21	3	7	7	1	21
24	8	8	24	4	6	4	2	24
48	8	8	48	12	4	4	1	48
63	8	8	63	9	7	0	1	63
64	10	8	80	8	8	2	1	66
12	9	9	12	3	4	8	6	12
15	9	9	15	3	5	8	4	15
16	9	9	16	2	8	8	2	16
18	9	9	18	6	3	6	4	18
18	9	9	18	9	2	9	4	18
24	9	9	24	3	8	8	1	24
26	9	9	26	13	2	0	3	26
28	9	9	28	4	7	5	2	28
78	9	9	78	13	6	0	1	78
80	9	9	80	10	8	0	1	80
12	10	10	12	4	3	9	8	12
14	10	10	14	2	7	8	6	14
14	10	10	14	7	2	6	7	14
18	10	10	18	2	9	9	2	18
27	10	10	27	3	9	9	1	27
32	10	10	32	4	8	6	2	32
75	10	10	75	15	5	5	1	75

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