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On The Generalized Divided Differences

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Abstract Let $V \subset \Re$ be a finite set and $f : \Re \to \Re$. $f(V)$ is divided difference of f at the points of V. The m-th order Peano derivative of $f(\{x\} \cup V)$ with respect to x is called generalized divided difference and is denoted by $f_m(x,V)$. The properties of $f_m(x,V)$ are studied.

Keywords: Peano derivative, Divided difference, Riemann* derivative, n-convex function.

MSC 2010: Primary 26A24, 26A99

1. INTRODUCTION

Let $f_m(x,V)$ be m-th order Peano derivative of $f(x_0, x_1 "\dots" x_n, x)$ regarded as a function of x. In [1] the authors have termed the iterated limit of divided difference by generalized divided difference and use it to study the properties of n-convex functions. In this article we have studied the properties of that generalized divided difference, which is equivalent to $f_m(x,V)$.

Let $f : E \to \Re,\ V = \{x_0, x_1, \dots, x_n\} \subset E$, then the divided difference of f at V is defined by

$$f(V) = f(x_0, \dots, x_n) = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)}$$

where

$$\omega(x) = \prod_{i=0}^{n} (x - x_i).$$

We write $f(x; V)$ instead of $f(V \cup \{x\})$.

Let $f : E \to \Re$, and $n \in \mathbb{N}$. Then f is n-convex in E if for each subset V of E containing n+1 points, $f(V) \geq 0$.

Let $x \in E - V$ *be* right hand limit point of E, then right Peano derivative of divided differences with respect to the set E is defined inductively as

$$f_1^+(x;V) = \lim_{\substack{y \to x^+ \\ y \in E}} \frac{f(y;V) - f(x;V)}{(y-x)}$$

and if $f_r^+(x;V)$ exist for $1 \leq r < m$, then m-th order derivative

$$f_m^+(x;V) = \lim_{\substack{y \to x^+ \\ y \in E}} \gamma_m(f, x, y, V)$$

where

$$\gamma_m(f, x, y, V) = \frac{m!}{(y-x)^m} \left\{ f(y;V) - \sum_{i=0}^{m-1} \frac{(y-x)^i}{i!} f_i^+(x;V) \right\}.$$

Here we assume $f_0^+(x;V) = f(x;V)$.

If $V = \varphi$, then we write $\gamma_m(f, x, y)$ instead of $\gamma_m(f, x, y, \varphi)$. In this case $f_m^+(x;V)$ is the usual right hand Peano derivative of f at x of order m and is denoted by $f_m^+(x)$. If $x \in E - V$ be a left hand limit point, we define left hand Peano derivative of divided difference $f_m^-(x;V)$ in similar way. If x is both sided limit point and $f_m^+(x;V)$ and $f_m^-(x;V)$ both exist and $f_i^+(x;V) = f_i^-(x;V)$ for $i = 1, 2, \ldots, m$, then f is said to have the m-th order Peano derivative of divided difference $f_m(x; V)$. Clearly for a fixed $\mathrm{V} \subset \mathrm{E}$, $f_m(x;V)$ is the m-th order Peano derivative of the function $f(x; V)$, regarded as a function of x. Hence for $x \notin V$, $f_m(x;V)$ exists if and only if $f_m(x)$ exists.

Let $f : E \to \Re$ and $V \subset E$ be finite and $x \in E - V$ be a limit point of E. Let $\{x_1, \ldots, x_k\}$ be k distinct points in $E - V$. Then as in [1]

$$[f, x, V]^k = \lim_{\substack{x_k \to x \\ x_k \in E}} \ldots \lim_{\substack{x_1 \to x \\ x_1 \in E}} f(x, x_1, \ldots, x_k; V).$$

If $V = \varphi$, then $[f, x]^k$ is called k-th order Riemann* derivative of f at x. In what follows from now we shall drop $y \in E$ under the limit notation.

2. PROPERTIES OF $f_n(x;V)$

Theorem 2.1. Let $f : E \to \Re$ be continuous, $V \subset E$ be finite set and $x \in E - V$ be right limit point of E. Now

$$\lim_{x_n \to x^+},\dots, \lim_{x_0 \to x^+} n!\, f(x_0, x_1, \dots, x_n; V)$$

exists if and only if $f_m^+(x;V)$ exists and they are equal.

Proof. We prove this by induction, for $n = 1$

$$\begin{aligned}\lim_{x_1 \to x^+} \lim_{x_0 \to x^+} f(x_0, x_1; V) &= \lim_{x_1 \to x^+} f(x, x_1; V)\\ &= \lim_{x_1 \to x^+} \frac{f(x;V) - f(x_1;V)}{x - x_1}\\ &= f_1^+(x;V).\end{aligned}$$

Suppose the theorem is true for $n = k$, $k \geq 1$. We prove the theorem for $n = k + 1$ and so the proof will be completed by induction.

So

$$\lim_{x_r \to x^+} \dots \lim_{x_0 \to x^+} r!\, f(x_0, x_1, \dots, x_r; V)$$

and $f_r^+(x;V)$ exists with equal value for $r = 1, 2, \dots, k$.

Now,

$$\lim_{x_{r+1} \to x^+} \dots \lim_{x_0 \to x^+} (r+1)!\, f(x_0, x_1, \dots, x_{r+1}; V) = \lim_{x_{r+1} \to x^+} (r+1) f_r^+(x; V \cup \{x_{r+1}\}) \quad (1)$$

As in Newton divided difference interpolation formula, we can write

$$\begin{aligned}f(u;V) &= f(x_0;V) + (u - x_0) f(x_0, x_1; V) + \dots\\ &\quad + (u - x_0)\dots(u - x_{r-1}) f(x_0, \dots, x_r; V)\\ &\quad + (u - x_0)\dots(u - x_r) f(u, x_0, \dots, x_r; V).\end{aligned}$$

Now performing the limit $\lim_{x_r \to x^+} \dots \lim_{x_0 \to x^+}$ on both sides we get

$$\begin{aligned}f(u;V) &= f(x;V) + (u - x) f_1^+(x;V) + \dots\\ &\quad \dots + \frac{(u-x)^r}{r!} f_r^+(x;V) + \frac{(u-x)^{r+1}}{r!} f_r^+(x; V \cup \{u\}).\end{aligned}$$

So

$$(r+1)f_r^+(x;V\cup\{u\})=\frac{f(u;V)-f^+(x;V)-\sum_{i=1}^{r}\frac{(u-x)^i}{i!}f_i^+(x;V)}{\frac{(u-x)^{r+1}}{(r+1)!}}$$

Putting $u=x_{r+1}$ and using (1) the proof follows

Remark 2.2. *T*heorem 2.1 also hold for left hand derivative. And for $V=\varphi$, it gives the well known result that Peano derivative and Riemann* derivative are equivalent.

Remark 2.3. *By* Theorem 2.1 $f_n^+(x;V)$ is same as $[f,x,V]^{(n,+)}$ *in* [1]

Theorem 2.4. Suppose $f:E\to\Re$ and $V\subset E$ be finite set and $x\in E-V$. Then $f_k(x;V)$ is divided difference of $f_k(x;\{t\})$ as a function of *t*.

Proof. As we define $f_n(x;V)$ is same as $[f,x,V]^{(n)}$ of [1]. So by Corollary 3.2 of [1] the result follows.

Lemma 2.5. *If* $f_k(x;V)$ *exists and if* $w\in V$. *Then*

$$f_k(x;V)=\frac{1}{w-x}\{kf_{k-1}(x;V)-f_k(x;V-\{w\})\}.$$

Proof. We prove this by induction, for $k=1$

$$\begin{aligned}f_1(x;V)&=\lim_{y\to x}f(x,y;V)\\&=\lim_{y\to x}\frac{f(x;V)-f(x,y;V-\{w\})}{w-x}\\&=\frac{f(x;V)-f_1(x;V-\{w\})}{w-x}\end{aligned}$$

So suppose, $f_i(x;V)=\frac{1}{w-x}\{if_{i-1}(x;V)-f_i(x;V-\{w\})\}$ for $i=1,2,\ldots,k-1$

Now,

$$\begin{aligned}f_k(x;V)&=\lim_{y\to x}\frac{f(y;V)-f(x;V)-\sum_{i=1}^{k-1}\frac{(y-x)^i}{i!}f_i(x;V)}{\frac{(y-x)^k}{k!}}\\&=\lim_{y\to x}\frac{(y-x)f(x,y;V)-\sum_{i=1}^{k-1}\frac{(y-x)^i}{i!}\frac{if_{i-1}(x;V)-f_i(x;V-\{w\})}{w-x}}{\frac{(y-x)^k}{k!}}\end{aligned}$$

$$= \lim_{y\to x} \frac{\frac{(y-x)}{(w-x)}\{f(y;V) - f(y,x;V-\{w\})\} - \sum_{i=1}^{k-1} \frac{(y-x)^i}{(i-1)!}\frac{f_{i-1}(x;V)}{w-x} + \sum_{i=1}^{k-1}\frac{(y-x)^i}{i!}\frac{f_i(x;V-\{x\})}{w-x}}{\frac{(y-x)^k}{k!}}$$

$$= \frac{k}{w-x}\lim_{y\to x}\frac{f(y;V) - \sum_{i=1}^{k-1}\frac{(y-x)^{i-1}}{(i-1)!}f_{i-1}(x;V)}{\frac{(y-x)^{k-1}}{(k-1)!}}$$

$$- \frac{1}{w-x}\lim_{y\to x}\frac{f(y;V-\{w\}) - f(x;V-\{w\}) - \sum_{i=1}^{k-1}\frac{(y-x)^i}{(i)!}f_i(x;V-\{w\})}{\frac{(y-x)^k}{k!}}$$

$$= \frac{1}{w-x}[kf_{k-1}(x;V) - f_k(x;V-\{w\})]$$

The proof is complete by induction.

Theorem 2.6. *Let V be a finite set and* $x_i \in V$ for $i = 0,1,\ldots,n.$. *Suppose* $f: E \to \Re$ *be such that* f_1 *exists on E. Then divided difference of* $f_1(x;V-\{x_0,...,x_n\})$ *at the points* $x_0, x_1, \ldots, x_n$, is

$$f_1(x_0,x_1,\ldots,x_n;V-\{x_0,x_1,\ldots,x_n\}) = \sum_{i=0}^{n} f_1(x_i;V-\{x_i\})$$

Proof. We prove it by induction. Suppose n=1 and by Lemma 2.5 we get

$$f_1(x_0;V-\{x_0\})=\frac{1}{x_1-x_0}\{f(x_0;V-\{x_0\})-f_1(x_0;V-\{x_0,x_1\})\}$$

$$f_1(x_1;V-\{x_1\})=\frac{1}{x_0-x_1}\{f(x_1;V-\{x_1\})-f_1(x_1;V-\{x_0,x_1\})\}$$

Since $f\left(x_0;V-\{x_0\}\right)=f\left(x_1;V-\{x_1\}\right)$

$$\begin{aligned} f_1(x_0;V-\{x_0\})+f_1(x_1;V-\{x_1\}) &= \frac{f_1(x_0;V-\{x_0,x_1\})-f_1(x_1;V-\{x_0,x_1\})}{x_0-x_1} \\ &= f_1\left(x_0,x_1;V-\{x_0,x_1\}\right) \end{aligned}$$

So suppose it is true for $(n-1), n\geq 2$. Hence

$$\begin{aligned} & f_1(x_0,x_1\ldots\ldots,x_n;V-\{x_0,x_1,\ldots,x_n\}) \\ &= \frac{f_1(x_0,\ldots,x_{n-1};V-\{x_0,x_1,\ldots,x_n\})-f_1(x_1,\ldots,x_n;V-\{x_0,x_1,\ldots,x_n\})}{x_0-x_n} \\ &= \frac{\sum_{i=0}^{n-1} f_1(x_i;V-\{x_i,x_n\})-\sum_{i=1}^{n} f_1(x_i;V-\{x_i,x_0\})}{x_0-x_n} \\ &= \frac{f_1(x_0;V-\{x_0,x_n\})-f_1(x_n;V-\{x_0,x_n\})}{x_0-x_n} \\ &+ \frac{\sum_{i=1}^{n-1}\left\{f_1(x_i;V-\{x_i,x_n\})-f_1(x_i;V-\{x_i,x_0\})\right\}}{x_0-x_n} \\ &= f_1(x_0,x_n;V-\{x_0,x_n\})+\sum_{i=1}^{n-1} f_1(x_i;V-\{x_i\}) \\ &= \sum_{i=0}^{n} f_1(x_i;V-\{x_i\}). \end{aligned}$$

Corollary 2.7. If f is *(n + 1)* convex on *E,* f_1 exists on *E;* then f_1 is *n* convex on *E.*

Proof. Suppose *f* is (*n*+1) convex then for any finite set $V=\{x_0,x_1,\ldots,x_n\}\subset E$ we have

$$\begin{aligned} f_1(x_0;\{x_1,\ldots,x_n\}) &= \lim_{y\to x_0}\frac{f(y,x_1,\ldots,x_n)-f(x_0,x_1,\ldots,x_n)}{(y-x_o)} \\ &= \lim_{y\to x_0} f(y,x_0,x_1,\ldots,x_n)\geq 0 \end{aligned}$$

Similarly, $f_1(x_i;V-\{x_i\}) \geq 0$ for all $i = 1, 2,..., n.$

Hence from Theorem 2.6

$$f_1(x_0, x_1, \ldots, x_n) \geq 0.$$

Therfore f_1 is n convex on E.

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