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Application of Combine Natural Transform and Adomian Decomposition Method in Volterra Integro-Differential Equations

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Abstract In this paper we introduced the combine Natural transform and Adomian decomposition method to solve the nonlinear voltrra integro-differential equations of first kind and second kind.To illustrate the method some examples are solved by using the above said method

Keywords: Volterra Integro-Differential Equations, Laplace Transform, Natural transform, Adomian Decomposition Method.

1. INTRODUCTION

1.1 Integral Equations

The volterra integro-differential equation is of the form[1], [2], [10].

$$u^{(i)}(x) = f(x) + \int_0^x K(x,t)F(u(t))dt \tag{1}$$

where $u^{(i)}(x) = \dfrac{d^i y}{dx^i}$ and initial conditions are given.

This paper deals with the nonlinear volterra integro-differential equation of the first kind and second kind. The standard form of nonlinear volterra integro-differential equation of second kind is given by [3]

$$u^{(i)}(x) = f(x) + \int_0^x K(x,t)F(u(t))dt \tag{2}$$

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where $u^{(i)}(x)=\frac{d^i y}{dx^i}, K(x,t)$ is the given kernel, *f*(*x*) is real valued function and F(u(x)) is non-linear function of u(x).

Similarly the non-linear voltera integro-differential equation of the first kind is given by[3]

$$\int_0^x K_1(x,t)F(u(t))dt+\int_0^x K_2(x,t)u^{(i)}(t)dt=f(x) \quad (3)$$

where $u^{(i)}(x)$ is the i^{th} derivative of $u(x)$, $K_1(x,t)$, $K_2(x,t)$ are the given kernels, $f(x)$ is real valued function and $F(u(x))$ is non-linear function of $u(x)$.

Various methods such as combine Laplace transform- Adomian decomposition method,iteration method,series solution method, ombine Sumudo transform-Adomian decomposition methods are used to solve such problems.

The advantage of this method is its capability of combining the two powerful methods for having the exact solution of such examples.The Voltera Integro-differential equations appeared in many physical application such as Newtons Diffusion and biological species coexisting together with increasing and decreasing rates of generating [3].

1.2 Natural Transform

The Natural transform initially defined by Khan and Khan[13] as N - transform who studied its properties and application as unsteady fluid flow problem over a plane wall. Belgacem et al.[4, 12] defined the inverse Natural transform,studied some properties and applications of Natural transforms. further applications of Natural transform can be seen in [6, 7, 11]

Laplace transform is the classical and extensively used integral transform which has many applications in various field such as physics, engineering and many more.The different applications of Laplace transform are seen in [5][8][9]. The Natural transform of the function $f(t)\in\Re^2$ is given by the following integral equation [4]

$$\mathbb{N}[f(t)]=R(s,u)=\int_0^\infty e^{-st}f(ut)dt \quad (4)$$

where $Re(s)>0, u\in(\tau_1,\tau_2)$ provided the function $f(t)\in\Re^2$ is defined in the set

$$\text{A}=[f(t)/\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{\frac{|t|}{\tau_j}}, \text{ if } \text{t}\in(-1)^j\times[0,\infty)]$$

The inverse Natural transform related with Bromwich contour integral [4, 12] is defined by

$$\mathbb{N}^{-1}[R(s,u)] = f(t) = \lim_{T\to\infty} \frac{1}{2\Pi i} \int_{\gamma - iT}^{\gamma + iT} e^{\frac{st}{u}} R(s,u) ds \tag{5}$$

1.3 Some Standard Result of Natural Transform

In this section we assume that all the considered functions are such that their Natural tranform exists.[4],[13]

1. $\mathbb{N}[1] = \frac{1}{s}$
2. $\mathbb{N}[t] = \frac{u}{s^2}$
3. $\mathbb{N}[t^n] = \frac{u}{s^2} n!$
4. $\mathbb{N}[e^{at}] = \frac{1}{s - au}$
5. $\mathbb{N}[\frac{sin(at)}{a}] = \frac{u}{s^2 + s^2u^2}$
6. $\mathbb{N}[cos(at)] = \frac{s}{s^2 + s^2u^2}$
7. $\mathbb{N}[\frac{t^{n-1}e^{at}}{(n-1)!}] = \frac{u^{n-1}}{(s-au)^2}$
8. $\mathbb{N}[f^{(n)}(t)] = \frac{s^n}{u^n}.R(s,u) - \sum_{n=0}^{\infty} \frac{S^{n-(k+1)}}{u^{n-k}}.u^{(k)}(0)$

   where $f^{(n)}(t) = \frac{d^n f}{dt^n}$

9. The Convolution Theorem

If $\{F(s,u)\}$ and G(s,u) are the Natural transforms of respective functions *f*(*t*) and $g(t)$ both defined in set A then, $\mathbb{N}[(f * g)] = u.F(s,u)G(s,u)$

2. MAIN RESULT (PART - I)

In this section we solve the non-linear voltera integro-differential equation of second kind by using the Combine Natural Transform and Adomian Decomposition Method.

2.1 The combine Natural transform and Adomian Decomposition Method

The standard form of nonlinear voltera integro -differential equation of second kind is given by[3]

$$u^{(n)}(x) = f(x) + \int_0^x K(x,t)F(u(t))dt \tag{6}$$

Here we consider the kernel K(x,t)as the difference kernel of the form $(x-t), e^{x-t}, \sin(x-t)$ etc.

$$u^{(n)}(x) = f(x) + \int_0^x K(x-t)F(u(t))dt \tag{7}$$

Apply Natural transform on both sides of the equation (7) and using the convolution theorem, we get

$$\mathbb{N}[u^{(n)}(x)] = \mathbb{N}[f(x)] + u.\mathbb{N}[k(x-t)].\mathbb{N}[F(u(x))] \tag{8}$$

$$\frac{s^n}{u^n}.R(s,u) - \sum_{n=0}^{\infty} \frac{s^{n-(k+1)}}{u^{n-k}}.u^{(k)}(0) = \mathbb{N}[f(x)] + u.\mathbb{N}[k(x-t)].\mathbb{N}[F(u(x))]$$

$$\frac{s^n}{u^n}.R(s,u) - \frac{s^{n-1}}{u^n}.u(0) - \frac{s^{n-2}}{u^{n-1}}u'(0) - \frac{s^{n-3}}{u^{n-2}}.u''(0) - ... - \frac{u^{(n-1)}(0)}{u} = \mathbb{N}[f(x)] + u.\mathbb{N}[k(x-t)].\mathbb{N}[F(u(x))]$$

$$\frac{s^n}{u^n}.R(s,u) = \frac{s^{n-1}}{u^n}.u(0) + \frac{s^{n-2}}{u^{n-1}}u'(0) + \frac{s^{n-3}}{u^{n-2}}.u''(0) + ... + \frac{u^{(n-1)}(0)}{u} + \mathbb{N}[f(x)] + u.\mathbb{N}[k(x-t)].\mathbb{N}[F(u(x))]$$

$$R(s,u) = \frac{u(0)}{s} + \frac{u}{s^2}u'(0) + \frac{u^2}{s^3}u''(0) + ... + \frac{u^{(n-1)}}{s^n}u^{(n-1)}(0) + \frac{u^n}{s^n}\mathbb{N}[f(x)] + \frac{u^{n+1}}{s^n}\mathbb{N}[k(x-t)].\mathbb{N}[F(u(x))]$$

Apply inverse Natural transform on both sides,we have

$$u(x) = u(0) + xu'(0) + ... + \frac{x^{n-1}}{(n-1)!}u^{(n-1)}(0) + \mathbb{N}^{-1}[\frac{u^n}{s^n}\mathbb{N}[f(x)]] + \mathbb{N}^{-1}[\frac{u^{n+1}}{s^n}\mathbb{N}[k(x-t)].\mathbb{N}[F(u(x))]] \tag{9}$$

Now to find the exact solution u(x), we apply adomian decomposition method for which consider

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{10}$$

where the components $u_n(x)$ can be calculated by using recursive relation. However the nonlinear terms $F((x))$ can be calculated by the adomian polynomials A_n in the form

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x) \tag{11}$$

$$where \quad A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(F\left(\sum_{n=0}^{\infty} \lambda^n U_n(x)\right)\right) n = 0,1,2... \tag{12}$$

$\therefore$ equation (9) becomes

$$\sum_{n=0}^{\infty} u_n(x) = u(0) + xu'(0) + ... + \frac{x^{n-1}}{(n-1)!}u^{(n-1)}(0) + \mathbb{N}^{-1}\left[\frac{u^n}{s^n}\mathbb{N}[f(x)]\right]$$
$$+\mathbb{N}^{-1}\left[\frac{u^{n+1}}{s^n}\mathbb{N}[k(x-t)\right].\mathbb{N}\left[\sum_{n=0}^{\infty} A_n(x)\right]$$

from this last equation we have the recursive relation as

$$u_0(x) = u(0) + xu'(0) + ... + \frac{x^{n-1}}{(n-1)!}u^{(n-1)}(0) + \mathbb{N}^{-1}\left[\frac{u^n}{s^n}\mathbb{N}[f(x)]\right] \tag{13}$$

$$u_{k+1}(x) = \mathbb{N}^{-1}\left[\frac{u^{n+1}}{s^n}\mathbb{N}[k(x-t)\right].\mathbb{N}\left[\sum_{n=0}^{\infty} A_n(x)\right], \qquad k \geq 0 \tag{14}$$

Now the equation (13) gives the value of A_0 which when substituted in equation (12) gives us the A_0 and from this value of A_0 we can easily calculate $u_1(x)$ using equation (14). In similar manner we can calculate $u_2(x)$, $u_3(x)$ and so on.

This gives us the series solution of given nonlinear integro-differential equation which may converges to exact solution provided that solution exists.

2.2 Illustrative Examples

Example (1) : Solve the nonlinear volterra integro-differential equation by using the combine Natural transform and Adomian Decomposition Method.

$$u'(x) = 1 - \frac{1}{3}e^x + \frac{1}{3}e^{-2x} + \int_0^x e^{(x-t)}u^2(t)dt \qquad u(0) = 0 \tag{15}$$

here the kernel k(x,t) = $e^{(x-t)}$ = difference kernel.
Solution :
Apply Natural transform on both sides of equation (18)

$$\mathbb{N}[u'(x)] = \mathbb{N}\left[1 - \frac{1}{3}e^{x} + \frac{1}{3}e^{-2x}\right] + \mathbb{N}[e^{(x-t)} * u^2(t)]$$
$$\frac{s}{u}R(s,u) - \frac{u(0)}{u} = \frac{1}{s} - \frac{1}{3}\frac{1}{s-u} + \frac{1}{3}\frac{1}{s+2u} + u.\frac{1}{s-u}\mathbb{N}[u^2(x)] \tag{16}$$

since u(0) = 0 we have

$$\frac{s}{u}R(s,u) = \frac{1}{s} - \frac{1}{3}\frac{1}{s-u} + \frac{1}{3}\frac{1}{s+2u} + u.\frac{1}{s-u}\mathbb{N}[u^2(x)]$$
$$\therefore R(s,u) = \frac{u}{s^2} - \frac{1}{3}\frac{u}{s(s-u)} + \frac{1}{3}\frac{u}{s(s+2u)} + \frac{u^2}{s(s-u)}\mathbb{N}[u^2(x)] \tag{17}$$

Apply inverse Natural transform on both sides of above equation,we get

$$u(x) = x - \frac{1}{3} - \frac{1}{3}e^{x} + \frac{1}{6} - \frac{1}{6}e^{-2x} + \mathbb{N}^{-1}[\frac{u^2}{s(s-u)}\mathbb{N}[u^2(x)]]$$
$$\therefore u(x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{8}x^4 + ... + \mathbb{N}^{-1}[\frac{u^2}{s(s-u)}\mathbb{N}[u^2(x)]] \tag{18}$$

Now apply the adomian decomposition method and using equations (11), (13), (14) we get

$$u_0(x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{8}x^4 + ... \tag{19}$$

$$u_{k+1}(x) = \mathbb{N}^{-1}\left[\frac{u^2}{s(s-u)}\mathbb{N}[A_k(x)]\right], \quad \text{for} \quad k \geq 0 \tag{20}$$

Now the adomian polynomials for $F(x) = u_2(x)$ are given by

$A_0(\mathrm{x}) = u_0^2$

$A_1(\mathrm{x}) = 2u_0u_1$ and so on...

substituting these values in the recursive relation (20) gives us

$$u_0(x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{8}x^4 + \dots \tag{21}$$

$$u_1(x) = \frac{1}{12}x^4 - \frac{1}{30}x^5 + \frac{1}{360}x^6 - \frac{1}{280}x^7 \dots \tag{22}$$

and so on...

Hence using the equation (10) the series solution of given nonlinear integro-differential equation (15) is given by

$$u(x) = x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \dots \tag{23}$$

which converges to the exact solution u(x) = 1− e^x.

Example (2) :

Solve the nonlinear volterra integro-differential equation by using the combine Natural transform and Adomian Decomposition Method.

$$\begin{aligned} u''(x) &= -\frac{5}{3}\sin x + \frac{1}{3}\sin 2x \\ &+ \int_0^x \cos(x-t)u^2(t)dt \quad u(0) = 0, u'(0) = 1 \end{aligned} \tag{24}$$

Solution : Apply Natural transform on both sides of equation (30)

$$\begin{aligned} \mathbb{N}[u''(x)] &= \mathbb{N}\left[-\frac{5}{3}\sin x + \frac{1}{3}\sin 2x\right] \\ &+ \mathbb{N}[\cos(x-t) * u^2(t)]\frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}u(0) - \frac{1}{u}u'(0) \\ &= -\frac{5}{3}\frac{u}{s^2+u^2} + \frac{1}{3}\frac{2u}{s^2+4u^2} + u.\frac{s}{s^2+u^2}\mathbb{N}[u^2(x)] \end{aligned} \tag{25}$$

since u(0) = 0 u’(0) = 1 we have

$$\begin{aligned} \therefore R(s,u) &= \frac{u}{s^2} - \frac{5}{3}\frac{u^3}{s^2(s^2+u^2)} \\ &+ \frac{1}{3}\frac{2u^3}{s^2(s^2+4u^2)} + \frac{u^3}{s(s^2+u^2)}\mathbb{N}[u^2(x)] \end{aligned} \tag{26}$$

Now apply inverse Natural transform on both sides we get

$$u(x) = x - \frac{5}{3}(x - \sin x) + \frac{1}{3}\left(\frac{1}{2}x - \frac{1}{4}\sin(2x)\right) + \mathbb{N}^{-1}\left[\frac{u^3}{s(s^2+u^2)}\mathbb{N}[u^2(x)]\right] u(x) = \frac{-1}{2}x + \frac{5}{3}\sin x - \frac{1}{12}\sin(2x) + \mathbb{N}^{-1}\left[\frac{u^3}{s(s^2+u^2)}\mathbb{N}[u^2(x)]\right] \tag{27}$$

on simplification of last equation, we have

$$u(x) = x - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{560}x^7 - ... + \mathbb{N}^{-1}\left[\frac{u^3}{s(s^2+u^2)}\mathbb{N}[u^2(x)]\right] \tag{28}$$

Now apply the adomian decomposition method and using equations (11)(13)(14) we get

$$u_0(x) = x - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{560}x^7 + ... \tag{29}$$

$$u_{k+1}(x) = \mathbb{N}^{-1}\left[\frac{u^3}{s(s^2+u^2)}\mathbb{N}[A_k(x)]\right], \qquad \text{for} \quad k \geq 0 \tag{30}$$

let the adomian polynomials are
$A_0 = u_0^2, A_1 = 2u_0u_1$ and so on we get

$$u_0(x) = x - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{560}x^7 + ... \tag{31}$$

$$u_1(x) = \frac{1}{60}x^5 - \frac{1}{504}x^7 + \frac{1}{12096}x^9 + \frac{41}{19958400}x^11 + ... \tag{32}$$

and so on...
Hence using the equation (10) the series solution of given nonlinear integro-differential equation (24) is given by

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + ... \tag{33}$$

which converges to the exact solution u(x) = sinx.

3. MAIN RESULT (PART - II)

In this section we solve the non-linear voltera integro-differential equation of first kind by using the Combine Natural Transform and Adomian Decomposition Method.

3.1 The combine Natural transform and Adomian Decomposition Method

The standard form of non-linear voltera integro-differential equation of first kind is given by[3]

$$\int_0^x K_1(x,t)F(u(t))dt + \int_0^x K_2(x,t)u^{(i)}(t)dt = f(x) \tag{34}$$

where $u^{(i)}(x)$ is the i^{th} derivative of $u(x)$, $K_1(x,t)$, $K_2(x, t)$ are the given kernels, $f(x)$ is real valued function and $F(u(x))$ is non-linear function of u(x).To determine the exact solution of given integro-differential equation the initial conditions should be given.

Now to solve such first kind equations consider the kernel K(x,t)as the difference kernel of the form $(x - t)$, e^{x-t}, $\sin(x - t)$ etc. so the given nonlinear volterra integro-differential equation is of the form

$$\int_0^x K_1(x-t)F(u(t))dt + \int_0^x K_2(x-t)u^{(n)}(t)dt = f(x) \tag{35}$$

Apply the Natural transform on both sides of the equation (36)and using the convolution theorem we have

$$u.\mathbb{N}[K_1(x-t)]\mathbb{N}[F(u(x))] + u.\mathbb{N}[K_2(x-t)]\mathbb{N}[u^{(n)}(t)] = \mathbb{N}[f(x)] \tag{36}$$

Let $\mathbb{N}[K_1(x-t)] = K_1(s,u), \mathbb{N}[K_2(x-t)] = K_2(s,u), \mathbb{N}[f(x)] = f(s,u)$

$$\begin{aligned}&\therefore u.K_2(s,u).\\&\left[\frac{s^n}{u^n}.R(s,u) - \frac{s^{n-1}}{u^n}.u(0) - \frac{s^{n-2}}{u^{n-1}}u'(0) - \frac{s^{n-3}}{u^{n-2}}.u''(0) - .... - \frac{u^{(n-1)}(0)}{u}\right]\\&= f(s,u) - u.K_1(s,u).\mathbb{N}[F(u(x))]\end{aligned} \tag{37}$$

$$\begin{aligned}\therefore R(s,u) = \frac{u(0)}{s} + \frac{u}{s^2}u'(0) + \frac{u^2}{s^3}u''(0) + ... + \frac{u^{n-1}}{s^n}u^{(n-1)}(0)\\ + \frac{u^{n-1}}{s^n}\frac{f(s,u)}{K_2(s,u)} - \frac{u^n}{s^n}\frac{K_1(s,u)}{K_2(s,u)}.\mathbb{N}[F(u(x))]\end{aligned} \tag{38}$$

Now apply inverse Natural transform on both sides,we get

$$u(x)=u(0)+xu'(0)+........+\frac{x^{n-1}}{(n-1)!}u^{(n-1)}(0)+\mathbb{N}^{-1}\left[\frac{u^{n-1}}{s^n}\frac{f(s,u)}{K_2(s,u)}\right] \\ -\mathbb{N}^{-1}[\frac{u^n}{s^n}\frac{K_1(s,u)}{K_2(s,u)}.\mathbb{N}[F(u(x))]] \tag{39}$$

Now to find the exact solution u(x), we apply adomian decomposition method for which consider

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \tag{40}$$

where the components $u_n(x)$ can be calculated by using recursive relation. However the nonlinear terms $F((x))$ can be calculated by the adomian polynomials A_n in the form

$$F(u(x))=\sum_{n=0}^{\infty}A_n(x) \tag{41}$$

$$where \quad A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left(F(\sum_{n=0}^{\infty}\lambda^n U_n(x))\right)n=0,1,2... \tag{42}$$

$\therefore$ equation (39) becomes

$$\sum_{n=0}^{\infty}u_n(x)=u(0)+xu'(0)+........+\frac{x^{n-1}}{(n-1)!}u^{(n-1)}(0)+\mathbb{N}^{-1}\left[\frac{u^{n-1}}{s^n}\frac{f(s,u)}{K_2(s,u)}\right] \\ -\mathbb{N}^{-1}\left[\frac{u^n}{s^n}\frac{K_1(s,u)}{K_2(s,u)}.\mathbb{N}\left[\sum_{n=0}^{\infty}A_n(x)\right]\right]$$

from this last equation we have the recursive relation as

$$u_0(x)=u(0)+xu'(0)+........ \\ +\frac{x^{n-1}}{(n-1)!}u^{(n-1)}(0)+\mathbb{N}^{-1}\left[\frac{u^{n-1}}{s^n}\frac{f(s,u)}{K_2(s,u)}\right] \tag{43}$$

$$u_{k+1}(x)=-\mathbb{N}^{-1}\left[\frac{u^n}{s^n}\frac{K_1(s,u)}{K_2(s,u)}.\mathbb{N}\left[\sum_{n=0}^{\infty}A_k(x)\right]\right],k\geq 0 \tag{44}$$

Now the equation (43) gives the value of u0 which when substituted in equation (42) gives us the A_0 and from this value of A_0 we can easily calculate $u_1(x)$ using equation (44). In similar manner we can calculate $u_2(x)$, $u_3(x)$ and so on.

This gives us the series solution of given nonlinear integro-differential equation which may converges to exact solution provided that solution exists.

3.2 Illustrative Examples

Example (1) : Solve the nonlinear volterra integro-differential equation by using the combine Natural transform and Adomian Decomposition Method.

$$\int_0^x e^{(x-t)}u^2(t)dt + \int_0^x e^{(x-t)}u'(t)dt = -1 + 3x.e^x + e^{2x}u(0) = 2 \quad (45)$$

Solution : Apply the Natural transform on both sides of the equation (45)and using the convolution theorem we have

$$u.\frac{1}{s-u}\mathbb{N}[u^2(x)] + u.\frac{1}{s-u}\mathbb{N}[u'(t)] = \mathbb{N}[-1 + 3x.e^x + e^{2x}]$$

$$\frac{u}{s-u}\mathbb{N}[u^2(x)] + \frac{u}{s-u}\left[\frac{s}{u}R(s,u) - \frac{u(0)}{u}\right] = -\frac{1}{s} + 3\frac{u}{(s-u)^2} + \frac{1}{s-2u} \quad (46)$$

since u(0) = 2 we have

$$R(s,u) = -\frac{3}{2}\frac{1}{s} + \frac{u}{s^2} + 3\frac{1}{s-u} + \frac{1}{2}\frac{1}{s-2u} - \frac{u}{s}\mathbb{N}[u^2(x)] \quad (47)$$

Apply inverse Natural transform on both sides, we get

$$u(x) = -\frac{3}{2} + x + 3e^x + \frac{1}{2}e^{2x} + \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}[u^2(x)]\right] \quad (48)$$

which on simplification gives,

$$u(x) = 2 + 5x + \frac{5}{2}x^2 + \frac{7}{6}x^3 + ... - \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}[u^2(x)]\right] \quad (49)$$

Now apply the adomian decomposition method and using equations (41),(43),(44) we get

$$u_0(x) = 2 + 5x + \frac{5}{2}x^2 + \frac{7}{6}x^3 + ... \quad (50)$$

$$u_{k+1}(x) = -\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}[A_k(x)]\right], \quad \text{for} \quad k \geq 0 \quad (51)$$

let the adomian polynomials are
$A_0 = u_0^2, A_1 = 2u_0u_1$ and so on we get

$$u_0(x) = 2 + 5x + \frac{5}{2}x^2 + \frac{7}{6}x^3 + ... \tag{52}$$

$$u_1(x) = -4x - 10x^2 - \frac{25}{3}x^3 - \frac{25}{4}x^4 - ... \tag{53}$$

$$u_2(x) = 8x^2 + \frac{80}{3}x^3 + \frac{65}{3}x^4 + ... \tag{54}$$

and so on...

Hence using the equation (40) the series solution of given nonlinear integro-differential equation (45) is given by

$$u(x) = 2 + x + \frac{1}{2!}x^2 + ... \tag{55}$$

$$u(x) = 1 + \left[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 .....\right] \tag{56}$$

which converges to the exact solution $u(x) = 1 + e^x$.
Example (2) : Solve the nonlinear volterra integro-differential equation by using the combine Natural transform and Adomian Decomposition Method.

$$\int_0^x (x-t)u^3(t)dt + \int_0^x (x-t)u''(t)dt = -\frac{10}{9} - \frac{4}{3}x + e^x + \frac{1}{9}e^{3x} u(0) = 1, u'(0) = 1. \tag{57}$$

Solution : Apply the Natural transform on both sides of the equation (56)and using the convolution theorem we have

$$u.\mathbb{N}[(x-t)] * u^3(x)] + u.\mathbb{N}[(x-t) * u''(t)] = \mathbb{N}\left[-\frac{10}{9} - \frac{4}{3}x + e^x + \frac{1}{9}e^{3x}\right] \tag{58}$$

$$u.\frac{u}{s^2}\mathbb{N}[u^3(x)] + u.\frac{u}{s^2}\left[\frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}u(0) - \frac{1}{u}u'(0)\right] = -\frac{10}{9}\frac{1}{s} - \frac{4}{3}\frac{u}{s^2} + \frac{1}{s-u} + \frac{1}{9}\frac{1}{s-3u}$$

since u(0) = 1, u'(0) = 1 we have

$$R(s,u) = -\frac{1}{9}\frac{1}{s} - \frac{1}{3}\frac{u}{s^2} + \frac{1}{s-u} + \frac{1}{9}\frac{1}{s-3u} - \frac{u^2}{s^2}\mathbb{N}[u^3(x)] \tag{59}$$

Apply inverse Natural transform on both sides, we get

$$u(x) = -\frac{1}{9} - \frac{1}{3}x + e^x + \frac{1}{9}e^{3x} - \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}[u^3(x)]\right] \tag{60}$$

which on simplification gives,

$$u(x) = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{5}{12}x^4 + \frac{7}{30}x^5 + \ldots\ldots - \mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}[u^3(x)]\right] \tag{61}$$

Now apply the adomian decomposition method and using equations (41),(43),(44) we get

$$u_0(x) = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{5}{12}x^4 + \frac{7}{30}x^5 + \ldots\ldots \tag{62}$$

$$u_{k+1}(x) = -\mathbb{N}^{-1}\left[\frac{u^2}{s^2}\mathbb{N}[A_k(x)]\right], \quad \text{for} \quad k \geq 0 \tag{63}$$

let the adomian polynomials are

$A_0 = u_0^3, A_1 = 3u_0^2u_1$ and so on we get

$$u_0(x) = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{5}{12}x^4 + \frac{7}{30}x^5 + \ldots\ldots \tag{64}$$

$$u_1(x) = -\frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{2}x^4 - \frac{7}{20}x^5 - \ldots\ldots \tag{65}$$

$$u_2(x) = \frac{1}{8}x^4 + \frac{9}{40}x^5 + \frac{1}{15}x^4 + \ldots \tag{66}$$

and so on...

Hence using the equation (40) the series solution of given nonlinear integro-differential equation (57) is given by

$$u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3\ldots\ldots \tag{67}$$

which converges to the exact solution $u(x) = e^x$.

CONCLUSION

In the present work, we have defined the combined method of Natural transform with Adomian decomposition method to solve the nonlinear integro-differential equations of the first kind and second kind. From the illustrative examples we can conclude that this method gives the results as accurate as possible.

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