Approximate Analytical Solution of Advection-Dispersion Equation By Means of OHAM

D J Prajapati* and N B Desai

1Government Engineering College, Modasa, 383 315, Gujarat, India
2A D Patel Institute of Technology, New Vidyanagar, 388 121, Gujarat, India
3Email: djganit@gmail.com

ARTICLE INFORMATION
Received: 02 July 2018
Revised: 15 July 2018
Accepted: 05 August 2018
Published online: September 6, 2018

ABSTRACT
This work deals with the analytical solution of advection dispersion equation arising in solute transport along unsteady groundwater flow in finite aquifer. A time dependent input source concentration is considered at the origin of the aquifer and it is assumed that the concentration gradient is zero at the other end of the aquifer. The optimal homotopy analysis method (OHAM) is used to obtain numerical and graphical representation.

Keywords: Advection, Dispersion, Convergence-control parameter, Discrete squared residual

DOI: https://doi.org/10.15415/mjis.2018.71003

1. Introduction
Solutions of advection-dispersion equation (ADE) may be used to predict the concentration of solutes in unsteady groundwater flow. Advection causes the contaminant plume to flow in the direction of groundwater water without any change in the shape. Dispersion of plume arises due to the variation in groundwater velocity. The heterogeneity of the porous medium is responsible for dispersion. The solute transport in heterogeneous aquifer is thus the combined process of advection and dispersion. The dispersion in the direction of groundwater flow is called longitudinal dispersion and the transverse dispersion is perpendicular to the groundwater flow direction. The ADE can be derived using Fick’s law and low of conservation of mass.

Analytical solutions in one-dimensional problems through semi-infinite or finite porous media have been presented by several researchers: (Mazaheri et. al. 2013, Kumar et. al. 2010, Marino et al. 1974, Singh et al. 2008) etc. The objective of this work is to derive an approximate analytical solution of ADE with the help of Optimal Homotopy Analysis Method (OHAM). In this work, an approximate analytical solution of one-dimensional ADE in heterogeneous finite aquifer is derived for continuous time dependent input source concentration of increasing nature.

2. Problem Formulation
We assume that the solute transport is primarily one-dimensional. The solutes are assumed to be conservative in the unsteady field. Consider an isotopic heterogeneous finite aquifer of length $L=1 \text{ km}$.

Let $C(x,t)$ be the contaminant concentration in the aquifer, $u$ the groundwater velocity and $D_l$ the longitudinal dispersion coefficient. Then the partial differential equation describing the one dimensional advection and dispersion is

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial x} \left( uC \right) = \frac{\partial}{\partial x} \left( D_l \frac{\partial C}{\partial x} \right)$$  

(1)

Expanding (1) we obtain

$$\frac{\partial C}{\partial t} + \left( u - \frac{\partial D_l}{\partial x} \right) \frac{\partial C}{\partial x} + C \frac{\partial u}{\partial x} = D_l \frac{\partial^2 C}{\partial x^2}$$

(2)

In heterogeneous medium, porosity changes with position. The velocity is non uniform as it depends on porosity. Hence the velocity of the flow field transporting the contaminants is considered spatially dependent. Let velocity at the origin
of the domain be \( u_0 \) which increases to \( u_0(1+b) \) at \( x = L \) where a real constant \( b < L \) ensures that the change in velocity is of small order i.e. the laminar condition of the flow is not affected (Kumar et al., 2010). The expression for velocity at any position \( x \) may be linearly interpolated as

\[
\nu(x) = \frac{x - L}{0 - L} u_0 + \frac{x - 0}{L - 0} u_0(1+b)
\]  

Simplifying we have

\[
\nu(x) = u_0(1+ax)
\]

where \( a = \frac{b}{L} \) is a constant less than 1 and serves as a heterogeneity parameter. Its different values represent media of varying heterogeneity.

Since mechanical dispersion depends on the flow, it is expected to increase with increasing flow speed. The dispersion parameter is considered to be proportional to the square of the velocity.

\[
D_t = D_0(1+ax)^2
\]

where \( D_0 = \alpha u_0^2 \) where \( \alpha \) is known as longitudinal dispersivity.

Using equation (4) and equation (5) in equation (2), we obtain

\[
\frac{\partial C}{\partial t} + (u_0 - 2aD_0)(1+ax)\frac{\partial C}{\partial x} + au_0C = D_0(1+ax)^2 \frac{\partial^2 C}{\partial x^2}
\]

which is linear advection-dispersion equation with variable coefficients. The solution \( C(x,t) \) of this equation represents the concentration of contaminants at any position and at any time.

The boundary condition at the origin is of uniform nature as defined by many researchers. It means that the input concentration at the origin of the medium (water bodies on the surface, groundwater level) and hence its source of pollution on the earth surface remains uniform at all times. But this is not the real situation at all times. Actually, the pollution at the surface and so the input will increase with time due to increasing human activities on the surface. So we have considered the inlet boundary condition as

\[
C(0,t) = t , \quad t > 0
\]

The concentration change is very negligible at the other end \( L=1 \) km of the aquifer. So the solute transport may not be affected at the other end of the aquifer. Hence we prescribe the outlet boundary condition as no flux boundary condition. i.e.

\[
\frac{\partial C}{\partial x}(1,t) = 0 \quad , \quad t > 0
\]

We solve equation (6) with boundary conditions equation (7) and equation (8) using optimal homotopy analysis method.

### 3. Solution of the Problem Using OHAM

To solve the problem by OHAM, we choose the initial guess of the solution \( C(x,t) \) as

\[
C_0(x,t) = t(e^{-x} + xe^{-1})
\]

which satisfies boundary conditions equation (7) and equation (8). The linear operator is selected as

\[
L[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial x^2}
\]

satisfying the property

\[
L[f] = 0 \quad \text{when} \quad f = 0.
\]

where \( 0 \leq q \leq 1 \) is the embedding parameter.

Further according to equation (6), the other operator is defined as

\[
N[\phi(x,t;q)] = D_0(1+ax)^2 \frac{\partial^2 \phi(x,t;q)}{\partial x^2} - au_0\phi(x,t;q)
\]

\[
-(u_0 - 2aD_0)(1+ax)\frac{\partial \phi(x,t;q)}{\partial x} - \frac{\partial \phi(x,t;q)}{\partial t}
\]

Let \( c_0 \) denote a nonzero auxiliary parameter. According to Liao[iii], the zeroth order deformation equation is

\[
(1-q)L[\phi(x,t;q) - C_0(x,t)] = c_0qH(x,t)N[\phi(x,t;q)]
\]

where \( H(x,t) \) is non-zero auxiliary function and \( \phi(x,t;q) \) is an unknown function.
Clearly, when \( q = 0 \) and \( q = 1 \), we have from equation (11) and equation (13),

\[
\phi(x,t;0) = C_0(x,t)
\]

and \( \phi(x,t;1) = C(x,t) \) (15)

Hence as \( q \) increases from 0 to 1, the solution \( \phi(x,t;q) \) deforms from the initial choice \( C_0(x,t) \) to the exact solution \( C(x,t) \) of equation (6). The determination of \( \phi(x,t;q) \) depends on the proper choices of the operator \( L \), the initial choice \( C_0(x,t) \) and the parameter \( c_0 \). We assume that all of them are properly chosen, the Maclaurin series

\[
\phi(x,t;q) = C_0(x,t) + \sum_{n=1}^{\infty} C_n(x,t)q^n
\]

exists and converges at \( q = 1 \). Hence we have the homotopy series solution

\[
C(x,t) = C_0(x,t) + \sum_{n=1}^{\infty} C_n(x,t)
\]

where \( C_n(x,t) = \left. \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m} \right|_{q=0} \) (18)

Differentiating equation (13) \( m \) times with respect to \( q \) and then dividing them by \( m! \) and then taking \( q = 0 \), we obtain high order deformation equations

\[
L[C_n(x,t) - \chi_n C_{m-1}(x,t)] = c_n H(x,t) R_n(x,t)
\]

subject to the boundary conditions

\[
C_n(0,t) = 0, \quad \frac{\partial C_n}{\partial x}(1,t) = 0, \quad m \geq 1
\]

where

\[
R_n(x,t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} L[\phi(x,t;q)]}{\partial q^{m-1}} \bigg|_{q=0}
\]

\[
= D_0(1+ax)^2 \frac{\partial^2 C_{m-1}}{\partial x^2} - (u_0 - 2aD_0)(1+ax) \frac{\partial C_{m-1}}{\partial x} - au_0 C_{m-1} - \frac{\partial C_{m-1}}{\partial t}
\]

\[
\chi_m = \begin{cases} 0 & \text{if } m \leq 1, \\ 1 & \text{if } m > 1. \end{cases}
\]

We assume \( H(x,t) = 1 \) for simplicity. Equations (19) are nonhomogeneous linear ordinary differential equations with constant coefficients for all \( m \geq 1 \). Solving equations (19)-(20) for \( m = 1 \), we obtain the first order homotopy approximation as

\[
C_1(x,t) = c_0 \left[ e^{-x} - \frac{x^2e^{-x}}{6} + \left( D_0e^{-1} - \frac{u_0e^{-1}}{2} \right)txe^{-x} + \left( u_0 + 2aD_0 + 2aD_0 + D_0 + u_0 \right)e^{-x} \right]
\]

\[
+ \left( D_0e^{-1} + 2u_0e^{-1} + 2au_0e^{-1} \right)tx - \left( 2a^2D_0 + 2aD_0 + D_0 + u_0 + au_0 \right)t
\]

Similarly the \( m \)th order homotopy approximation \( C_m(x,t) \) can be obtained for successive values of \( m \). The convergence of the \( m \)th order homotopy approximation depends on the proper choice of \( c_0 \). The homotopy series solution can then be written as

\[
C(x,t) = C_0(x,t) + C_1(x,t) + \ldots
\]

The solution represents solute or contaminant concentration at distance \( x \) for any time \( t \) whose convergence depends on the proper choice of the convergence-control parameter \( c_0 \). To select proper value of \( c_0 \), we use the discrete squared residual at the \( m \)th order homotopy approximation denoted by \( E_m \) and defined by (Liao et al. 2010) as

\[
E_m = \frac{1}{(M+1)(N+1)} \sum_{i=1}^{M} \sum_{j=1}^{N} \left[ \sum_{m=0}^{M} C_m \left( \frac{i}{M}, \frac{j}{N} \right) \right]^2
\]

As fast the residual \( E_m \) decreases, the accuracy of the corresponding homotopy approximation increases. At the \( m \)th order approximation, the corresponding optimal value of the convergence-control parameter \( c_0 \) is given by the minimum of \( E_m \) corresponding to a nonlinear algebraic equation to be solved from
This approach for obtaining the optimal value of $c_0$ has been used for solving a number of problems for nonlinear ordinary and partial differential equations by many researchers (Liao, 2010, Prajapati et al. 2015, 2016, 2017, Liao et al. 2012, 2013, Vejravelu et al., 2012). We have obtained the solution up to 10th order approximation. The optimal value of $c_0$ is found by the minimum of $E_{10}$ using Mathematica. Here $E_{10}$ attains its minimum value $5.23828 \times 10^{-7}$ at $c_0 = -2.42025$ which we can observe in Figure 1 also. We take $M = N = 50$ for finding $E_{10}$.

![Figure 1](image.png)

**Fig 1** The discrete squared residual at 10th order homotopy approximation

Table 1. The discrete squared residual of governing equation (6) by means of $c_0 = -2.42025$

<table>
<thead>
<tr>
<th>Order of approximation $m$</th>
<th>Discrete Squared Residual $E_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.24554E-1</td>
</tr>
<tr>
<td>4</td>
<td>3.53453E-3</td>
</tr>
<tr>
<td>6</td>
<td>2.38709E-4</td>
</tr>
<tr>
<td>8</td>
<td>9.68075E-6</td>
</tr>
<tr>
<td>10</td>
<td>4.7776E-7</td>
</tr>
</tbody>
</table>

From the Table 1, we can see that the squared residual decreases as we increase the order of approximation. So the 10th order homotopy approximation is accurate.

### 4. Numerical and Graphical Representation

BVPh, a Mathematica package, is used to obtain numerical values of concentration. Table 2 indicates the numerical values of the solute concentration

Table 2. Numerical values of the solute concentration $C(x, t)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$t=1$</th>
<th>$t=1.1$</th>
<th>$t=1.2$</th>
<th>$t=1.3$</th>
<th>$t=1.4$</th>
<th>$t=1.5$</th>
<th>$t=1.6$</th>
<th>$t=1.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.00000</td>
<td>1.10000</td>
<td>1.20000</td>
<td>1.30000</td>
<td>1.40000</td>
<td>1.50000</td>
<td>1.60000</td>
<td>1.70000</td>
</tr>
<tr>
<td>0.01</td>
<td>0.96338</td>
<td>1.06334</td>
<td>1.16330</td>
<td>1.26326</td>
<td>1.36322</td>
<td>1.46318</td>
<td>1.56314</td>
<td>1.66310</td>
</tr>
<tr>
<td>0.02</td>
<td>0.92713</td>
<td>1.02705</td>
<td>1.12697</td>
<td>1.22689</td>
<td>1.32681</td>
<td>1.42673</td>
<td>1.52665</td>
<td>1.62657</td>
</tr>
<tr>
<td>0.03</td>
<td>0.89124</td>
<td>0.99112</td>
<td>1.09100</td>
<td>1.19088</td>
<td>1.29076</td>
<td>1.39064</td>
<td>1.49052</td>
<td>1.59040</td>
</tr>
</tbody>
</table>

$C(x, t)$ [kg/(km)$^3$] for different distance $x$ (km) and time $t$ (year) up to 10th order approximation using $c_0 = -2.42025$. We have considered

$L = 1(km)$, $a = 0.1(km)^{-1}$, $u_0 = 0.11\left(\frac{km}{yr}\right)$ and $D_0 = 0.21\left(\frac{(km)^2}{yr}\right)$

to obtain numerical values of the solution.

Figure 2 shows the concentration profiles of the contaminant for fixed values of times. Numerical values of Table 2 are used for Figure 2. Figure 3 shows the graph of concentration for fixed positions. We use numerical values of Table 2 for Figure 3 also.
5. Conclusion

The approximate analytical solution of advection-dispersion equation with variable coefficients is obtained by optimal homotopy analysis method with time-dependent input source concentration. The contaminant transport behaves as expected i.e. the concentration at a fixed time decreases as the distance increases and the concentration at a fixed position advances with time. The solution is useful as a preliminary predictive tool for simulating the solute migration in aquifer due to the release of a time-dependent source.
References


Books


