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On The Generalized Natural Transform

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ABSTRACT

In this paper, we introduce the Natural transform in the generalized sense with the help of distribution theory. Inversion, Uniqueness theorems and some properties of generalized in¬tegral transform are proved.

1. Introduction

Integral transform method has wide range of applications in the various fields of science and engineering. In most of the cases the physical phenomenon is converted into an ordinary differential equations and partial differential equations which can be solved by integral transform method. This is the basic thing by which the researchers are being motivated to define new integral transforms and used to solve many problems in the field of applied mathematics. Recently, the new integral transform Natural transform (N-transform) was introduced by (Khan and Khan, 2008) and studied its properties and some applications. Later on (Silambarasan *et. al.*, 2011 and Belgacem *et. al.*, 2012) defined the inverse Natural transform and studied some properties and applications of Natural transforms.

The distribution theory provides powerful analytical technique to solve many problems that arises in the applied field. This gives rise to define the various integral transforms to the distribution space (Lookner, 2010, 2012 & 2013, Omari, 2014, Shah, 2015, Pathak, 1997, Schwartz, 1950, 51 and Zemanian, 1987).The aim of this paper is to extend the Natural transform in the distributional space of compact support and to investigate some properties and theorems of the generalized integral transform.

1.1. The Natural Transformation

The Natural transform of the function $f(t)\in\Re^2$ is sectionwise continuous, exponential order and defined in the set

$$A=\left[f(t)/\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{\frac{|t|}{\tau_j}},if\ t\in(-1)^j\times[0,\infty]\right]$$

is denoted by symbol $\mathbb{N}[f(t)]=R(s,u)=\frac{1}{u}\int_0^\infty e^{-\frac{st}{u}}f(t)dt$ where s and u are the transform variables and is defined by an integral equation (Belgacem *et. al.*, 2012)

$$\mathbb{N}[f(t)]=R(s,u)=\int_0^\infty e^{-st}f(ut)\,dt \quad (1)$$

where $\operatorname{Re}(s)>0,u\in(\tau_1,\tau_2)$ The above equation can be written as

$$\mathbb{N}[f(t)]=R(s,u)=\frac{1}{u}\int_0^\infty e^{-\frac{st}{u}}f(t)dt \quad (2)$$

The inverse Natural transform of function $R(s, u)$ is denoted by symbol $\mathbb{N}^{-1}[R(s,u)]=f(t)$ and is defined with Bromwich contour integral (Silambarasan *et. al.*, 2011 and Belgacem *et. al.*, 2012)

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$$\mathbb{N}^{-1}\left[R(s,u)\right]=f(t)=\lim_{T\to\infty}\frac{1}{2\Pi i}\int_{\gamma-iT}^{\gamma+iT}e^{\frac{st}{u}}R(s,u)\,ds \qquad (3)$$

If R(s,u) is the Natural transform, F(s) is the Laplace transform and G(u)is Sumudo transform of function f(t) $\in$ A then we can have Natural-Laplace and Natural-Sumudo duality as

$$\mathbb{N}\left[f(t)\right]=R(s,u)=\int_0^\infty e^{-st}f(ut)\,dt=\frac{1}{u}F\left(\frac{s}{u}\right) \qquad (4)$$

and

$$\mathbb{N}\left[f(t)\right]=R(s,u)=\int_0^\infty e^{-st}f(ut)\,dt=\frac{1}{s}G\left(\frac{u}{s}\right) \qquad (5)$$

We can extract the Laplace, Sumudu, Fourier and Mellin transform from Natural transform and which shows that Natural transform convergence to Laplace and Sumudu transform (Shah *et. al.*, 2015). Moreover Natural transform plays as a source for other transform and it is the theoretical dual of Laplace transform. Further study and applications of Natural transform can be seen in (Silambarasan *et. al.*, 2012, Lookner *et. al.*, 2013, Chindhe, *et. al.*, 2016 & 2017).

1.2. Basic Properties of Natural Transform

1. $\mathbb{N}[1]=\frac{1}{s}$
2. $\mathbb{N}\left[t^n\right]=\frac{u^n}{s^{n+1}}n!$
3. $\mathbb{N}\left[e^{at}\right]=\frac{1}{s-au}$
4. $\mathbb{N}\left[\frac{t^{n-1}e^{at}}{(n-1)!}\right]=\frac{u^{n-1}}{(s-au)^2}$
5. $\mathbb{N}\left[f^{(n)}(t)\right]=\frac{s^n}{u^n}.R(s,u)-\sum_{n=0}^{\infty}\frac{s^{n-(k+1)}}{u^{n-k}}.u^{(k)}(0)$
   where $f^{(n)}(t)=\frac{d^nf}{dt^n}$
6. If F(s, u) and G(s,u) are the Natural transforms of respective functions f(t) and g(t) both defined in set A then $\mathbb{N}\left[(f*g)\right]=u.F(s,u)G(s,u)$

2. Generalized Natural Transform

The author Deshna Loonkar (Lookner *et. al.*, 2013) have studied distributional Natural transform and motivated from that study, here we construct the testing function space to define the generalized integral transform and prove some theorems of generalized Natural transform.

2.1. Testing function space $\mathcal{D}_{a,b}$

Let $\mathcal{D}_{a,b}$ denotes the space of all complex valued smooth functions $\phi(t)$on $-\infty<t<\infty$ which the functions $\gamma_k(\phi)$ defined by

$$\gamma_k(\phi)\triangleq\gamma_{a,b,k}(\phi)\triangleq \sup_{\infty<t<\infty}\left|K_{a,b}(t)D^k(t)\right|<\infty \qquad (6)$$

Where

$$K_{a,b}(t)=\begin{cases}e^{at} & 0\le t<\infty\\ e^{bt} & -\infty<t<0.\end{cases}$$

This $\mathcal{D}_{a,b}$ is linear space under the pointwise addition of function and their multiplication by complex numbers. Each γ_k is clearly a seminorm on $\mathcal{D}_{a,b}$ and γ_0 is a norm. We assign the topology generated by the sequence of seminorm $(\gamma_k)_{k=0}^{\infty}$ there by making it a countably multinormed space. Note that for each fixed s and u the kernel $\frac{1}{u}e^{\frac{-st}{u}}$ as a function of t is a member of $\mathcal{D}_{a,b}$ iff $a<\operatorname{Re}\left(\frac{s}{u}\right)<b$. With the usual argument we can show that $\mathcal{D}_{a,b}$ is complete and hence a Frechet space. $\mathcal{D}'_{a,b}$ denotes the dual of $\mathcal{D}_{a,b}$ i.e. f is member of $\mathcal{D}'_{a,b}$ iff it is continuous linear function on $\mathcal{D}_{a,b}$. Thus $\mathcal{D}'_{a,b}$ is a space of generalized functions.

Now we define the generalized Natural Transform. Given a generalized Natural transformable generalized function f, the strip of definition Ω_f for $\mathbb{N}[f]$ is a set in $\mathbb{C}$ defined by $\Omega_f\triangleq\left\{(s,u):\omega_1<\operatorname{Re}\left(\frac{s}{u}\right)<\omega_2\right\}$ since f or each $(s,u)\in\Omega_f$ the kernel $e^{\frac{-st}{u}}$ as a function of t is a member of $\mathcal{D}'_{\omega_1,\omega_2}$.

For $f\in\mathcal{D}'_{\omega_1,\omega_2}$, we can define the generalized Natural transform of f as conventional function

$$R_f(s,u)\triangleq\mathbb{N}\left[f(t)\right]\triangleq\left<f(t),\frac{1}{u}e^{\frac{-st}{u}}\right> \qquad (7)$$

We call Ω_f the region (or strip) of definition for $\mathbb{N}\left[f(t)\right]$ and w_1 and w_2 the abscissas of definition. Note that the properties like linearity and continuity of generalized Natural transform will follows from (Zemanian *et. al.*, 1987)

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Theorem 2.1 If $R_f(s,u) \triangleq \mathbb{N}[f(t)]$ for $(s,u) \in \Omega_f$ then $R_f(s,u)$ is analytic on Ω_f

Proof: Let (s, u) be arbitrary but fixed point in $\Omega_f = \left\{(u,s)\omega_1 < \operatorname{Re}\left(\frac{s}{u}\right) < \omega_2\right\}$.

Choose the real positive number a,b and r such that $\omega_1 < a\operatorname{Re}\left(\frac{s}{u} - r\right) < \operatorname{Re}\left(\frac{s}{u} + r\right) < b < \omega_2$.

Let ΔS be the complex increment such that $|\Delta S| < r$ and as $\Delta S \neq 0$ we have

$$\frac{R_f(s+\Delta S,u) - R_f(s,u)}{\Delta S} - < f(t), \frac{\partial}{\partial s}\frac{1}{u}e^{\frac{-st}{u}} > \\ = < f(t), \psi\Delta S(t) > \quad (8)$$

Where $\psi\Delta S(t) = \frac{1}{\Delta S}\left[e^{\frac{-(s+\Delta S)t}{u}} - e^{\frac{-st}{u}}\right] - \frac{\partial}{\partial s}\frac{1}{u}e^{\frac{-st}{u}}$

Note that $\psi\Delta S \in \mathcal{D}_{a,b}$ so that equation (8) is meaningful. We shall now show that as $|\Delta S| \to 0, \psi\Delta S(t)$ converges to zero in $\Re'_{a,b}$. Since $f \in \Re'_{a,b}$ this will imply that $< f(t), \psi\Delta S(t) > \to 0$. From equation(8)and choose a close to ω_1 and b close to ω_2 which gives the analyticity of $R_f(s,u)$ on Ω_f.

Let C denotes the circle with center at $\frac{s}{u}$ and radius r1 where $0 < r < r_1 < \min\left(\operatorname{Re}\left(\frac{s}{u} - a, b - \operatorname{Re}\left(\frac{s}{u}\right)\right)\right)$. We may interchange differentiation on s with differentiation on t and using Cauchy integral formula so that equation (8) becomes

$$(-D_t)^k \psi\Delta S(t) = \frac{1}{\Delta S}\left[\left(\frac{s+\Delta S}{u}\right)^k e^{\frac{-(s+\Delta S)t}{u}} - \left(\frac{s}{u}\right)^k e^{\frac{-st}{u}}\right] - \frac{\partial}{\partial s}\left(\frac{s}{u}\right)^k \frac{1}{u}e^{\frac{-st}{u}}$$

$$= \frac{1}{\Delta S 2\Pi i}\int_C \left(\frac{1}{\xi - \left(\frac{s+\Delta S}{u}\right)} - \frac{1}{\xi - \left(\frac{s}{u}\right)}\xi^k e^{\frac{-\xi t}{u}} d\xi - \frac{1}{2\Pi i}\int_C \frac{\xi^k e^{\frac{-\xi t}{u}}}{\xi - \left(\frac{s}{u}\right)^2} d\xi\right)$$

$$= \frac{\Delta S}{2\Pi i}\int_C \frac{\xi^k e^{\frac{-\xi t}{u}}}{\xi - \left(\frac{s+\Delta S}{u}\right)\xi - \left(\frac{s}{u}\right)^2} d\xi$$

Now for all $\xi \in C$ and $-\infty < t < \infty, \left|K_{a,b}(t)\xi^k e^{\frac{-\xi t}{u}}\right| \leq M$ where M is constant independent of ξ and t. Moreover $\xi - \left(\frac{s+\Delta s}{u}\right) > r_1 - r > 0$ and $\left|\xi - \left(\frac{s}{u}\right)\right| = r_1$

$$\left|K_{a,b}(t)D^k(t)\psi\Delta S\right| \leq \frac{|\Delta S|}{2\Pi}\int_C \frac{M}{(r_1 - r)r_1^2}|d\xi| \\ \leq \frac{|\Delta S| M}{(r_1 - r)r_1^2}$$

The RHS is independent of t and converges to zero as $|\Delta S| \to 0$. This shows that $\psi_{\Delta S}$ converges to zero in $\mathcal{D}_{a,b}$ as $|\Delta S| \to 0$ which completes the proof of theorem. Similar proof can be made for the another variable u.

Theorem 2.2 [Characteriztion Theroem]

The necessary condition for the function *Rf(s,u)* to be the Natural transform of generalized function f are that *Rf(s,u)* is analytic on Ω_f and for each closed strip $\left\{(u,s) : a \leq \operatorname{Re}\left(\frac{s}{u}\right) \leq b\right\}$ of Ω_f there be a polynomial such that $|R_f(s,u)| \leq \frac{1}{u}P\left(\left|\frac{s}{u}\right|\right)$ for $a \leq \operatorname{Re}\left(\frac{s}{u}\right) \leq b$. The polynomial P will depend in general on a and b.

Proof: The analyticity of $R_f(s,u)$ has been already proved in the previous theorem. By the definition of the Natural transform, f is a member of $\mathcal{D}'_{a.b}$ where $\omega_1 < a < b < \omega_2$ so that there exists a constant M and non-negative integer r such that for $a \leq \operatorname{Re}\left(\frac{s}{u}\right) \leq b$

$$\left|R_f(s,u) = < f(t), \frac{1}{u}e^{\frac{-st}{u}} >\right| \\ \leq \frac{1}{|u|} M \underset{0\leq k\leq r}{Max} \sup_t |K_{a,b}(t)D_t^k e^{\frac{-st}{u}}| \\ \leq \frac{1}{|u|} M \underset{0\leq k\leq r}{Max}\left|\frac{s}{u}\right|^k \sup_t \left|K_{a,b}(t)D_t^k e^{\frac{-st}{u}}\right| \\ \leq \frac{1}{|u|}P\left(\left|\frac{s}{u}\right|\right)$$

This polynomial $P\left(\left|\frac{s}{u}\right|\right)$ depends in general on the choices of a and b.

3. Inversion and Convolution

Now for the generalized inversion formula for Natural transform we require following two lemma which can be easily proved by (Zemanian *et. al*, 1987)

Lemma A: Let $R_f(s,u)=\mathbb{N}[f(t)]$ for $\omega_1\le \operatorname{Re}\left(\frac{s}{u}\right)\le\omega_2$ and let $\phi\in\mathcal{D}$, set $\psi(s,u)=\frac{1}{u}\int_{-\infty}^{\infty}\phi(t)e^{\frac{st}{u}}dt$ Then for any fixed real number p with $0<p<\infty$

$$\frac{1}{u}\int_{-p}^{p}<f(\tau),\frac{1}{u}e^{\frac{-s\tau}{u}}>\psi(s,u)d\omega \\ =<f(\tau),\frac{1}{u}\int_{-p}^{p}e^{\frac{-s\tau}{u}}\psi(s,u)d\omega> \tag{9}$$

Where $s=\sigma+i\omega$ and σ is fixed with $\sigma_1<\sigma<\sigma_2$

Lemma B: Let a,b, a,b,σ and r be real numbers with $a<\sigma<b$. Also let $\phi\in\mathcal{D}$ then

$$\frac{1}{\Pi u}\int_{-\infty}^{\infty}\phi(t+\tau)e^{\frac{\sigma t}{u}}\frac{\sin(rt)}{t}dt \tag{10}$$

converges in $\mathcal{D}_{a,b}$ to $\phi(\tau)$ as $r\to\infty$

3.1. Inversion Theorem

Let $R_f(s,u)=\mathbb{N}[f(t)]$ for $r_1<\operatorname{Re}\left(\frac{s}{u}\right)<r_2$ be any real variable then in the sense of convergence in $\mathcal{D}'$

$$f(t)=\lim_{p\to\infty}\frac{1}{2\Pi i}\int_{r-ip}^{r+ip}R_f(s,u)e^{\frac{st}{u}}ds \tag{11}$$

where r is fixed number such that $r_1<r<r_2\ (s=r+i\omega)$

Proof : Let $\phi\in\mathcal{D}$, let us choose two real numbers a and b such that $r_1<a<r<b<r_2$. To prove the theorem it is sufficient to prove that

$$\lim_{p\to\infty}<\frac{1}{2\Pi i}\int_{r-ip}^{r+ip}R_f(s,u)e^{\frac{st}{u}}ds,\phi(t)>=<f(t),\phi(t)> \tag{12}$$

Now the integral on s is contineous function of t and therefor the left hand side of above equation can be written as

$$\frac{1}{2\Pi}\int_{-\infty}^{\infty}\phi(t)\int_{-p}^{p}R_f(s,u)e^{\frac{st}{u}}d\omega dt \tag{13}$$

where $s=r+i\omega$ and $p>0$

since $\phi(t)$ is of bounded support and the integrand is a contineous function of (t,ω) the order of integration may be changed

$$\frac{1}{2\Pi}\int_{-p}^{p}R_f(s,u)\int_{-\infty}^{\infty}\phi(t)e^{\frac{st}{u}}dtd\omega$$

$$\frac{1}{2\Pi}\int_{-p}^{p}\frac{1}{u}<f(\tau),e^{\frac{-s\tau}{u}}>\int_{-\infty}^{\infty}\phi(t)e^{\frac{st}{u}}dtd\omega$$

by using the above lemma, we have

$$<f(\tau),\frac{1}{2\Pi u}\int_{-p}^{p}e^{\frac{-s\tau}{u}}\int_{-\infty}^{\infty}\phi(t)e^{\frac{st}{u}}dtd\omega>$$

the order of integration for the repeated integral here in may be changed because again $\phi(t)$ is of bounded support and the integrand is a contineous function of (t,ω) then we have

$$<f(\tau),\frac{1}{2\Pi u}\int_{-\infty}^{\infty}\phi(t)\int_{-p}^{p}e^{\frac{s(t-\tau)}{u}}d\omega dt>$$
$$<f(\tau),\frac{1}{\Pi u}\int_{-\infty}^{\infty}\phi(t+\tau)\frac{\sin(pt)}{t}e^{\frac{rt}{u}}dt>$$

But the last expression tends to $<f(t),\phi(t)>$ as $r\to\infty$ due to lemma B. Hence the proof.

3.2 Convolution

Definition:- For two generalized functions f and g in $\Re_{a,b}(a,\le b)$ the convolution product f * g is defined by expression,

$$<f*g,\phi>\triangleq<f(t),<g(\lambda),\phi(\lambda+t)>>\quad \phi\in\Re_{a,b} \tag{14}$$

Theorem Let f and g be two generalized functions such that $R_f(s,u)=\mathbb{N}[f(t)]$ and $R_g(s,u)=\mathbb{N}[g(t)]$ then Natural transform of the convolution is given by $\mathbb{N}[f*g]=uR_f(s,u)R_g(s,u)$

Proof: By the definition of convolution we have

$$\mathbb{N}[f*g]=\frac{1}{u}<(f*g)(t),e^{\frac{-st}{u}}>$$
$$=\frac{1}{u}<f(t),<g(\lambda),e^{\frac{-s(\lambda+t)}{u}}>>$$
$$=\frac{1}{u}<f(t),e^{\frac{-st}{u}}><g(\lambda),e^{\frac{-s\lambda}{u}}>$$

$$= u. < f(t), \frac{1}{u} e^{\frac{-st}{u}} > < g(\lambda), \frac{1}{u} e^{\frac{-s\lambda}{u}} >$$
$$= u.R_f(s,u) R_g(s,u)$$

Hence the proof.

Conclusion

In this paper we extended the Natural transform in the distributional space of compact support and so defined generalized Natural transform. The analyticity theorem and inversion theorem are proved. This paper might be a new window for the researcher to study of generalized integral transforms.

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