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Fixed Points of Almost Generalized (α, ψ)-Contractions with Rational Expressions

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Abstract In this paper, we introduce almost generalized (α, ψ)-contractions with rational expression type mappings and establish the existence of fixed points for such mappings in complete partially ordered metric spaces. Further, we define `Condition (H)' and prove the existence of unique fixed point under the additional assumption `Condition (H)'. Our results generalize the results of Arshad, Karapinar and Ahmad [1] and Harjani, Lopez and Sadarangani [2].

Keywords: α-admissible; (α, ψ)-contraction mapping; generalized (α, ψ)-contraction mapping; almost Jaggi contraction; almost generalized (α, ψ)-contraction map with rational expression.

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1. INTRODUCTION

Generalization of contraction conditions and proving the existence of fixed points is an interesting aspect. Recently Samet, Vetro and Vetro [4] introduced a new concept namely (α,ψ)-contraction mappings which generalize contraction mappings and proved the existence of fixed points of such mappings in metric space setting. After that Karapinar and Samet [5] introduced generalized (α,ψ)-contraction mappings and proved fixed point results and its extension to partially ordered metric spaces can be found in [6]. In this direction, we introduce almost generalized (α,ψ)-contractions with rational expression type mappings and establish the existence of fixed points for such mappings in complete partially ordered metric spaces. Further, we define 'Condition (H)' and prove the uniqueness of fixed point under the additional assumption 'Condition (H)'. Our results generalize the results of Arshad, Karapinar and Ahmad [1] and Harjani, Lopez and Sadarangani [2].

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In the following, Ψ denotes the family of non-decreasing functions $\psi:[0,\infty)\to[0,\infty)$ such that ψ is continuous on $[0,\infty)$ and $\sum_{n=1}^{\infty}\psi^n(t)<+\infty$ for each t > 0, where ψ^n is the n^{th} iterate of ψ.

Remark 1.1. Any function $\psi\in\Psi$ satisfies $\lim_{n\to\infty}\psi^n(t)=0$ and $\psi(t)<t$ for any t > 0.

Definition 1.2. [4] Let (X,d) be a metric space, $T:X\to X$ and $\alpha:X\times X\to[0,\infty)$. We say that T is α-*admissible*, if $x,y\in X$, $\alpha(x,y)\geq 1\Rightarrow\alpha(Tx,Ty)\geq 1$. (1.2.1)

Definition 1.3. [4] Let (X,d) be a metric space and $T:X\to X$ be a selfmap of X. If there exist two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that $\alpha(x,y)d(Tx,Ty)\leq\psi(d(x,y))$ for all $x,y\in X$, then we say that T is a (α,ψ)*-contraction mapping.*

Remark 1.4. If $T:X\to X$ satisfies the Banach contraction principle, then T is an (α,ψ)-contraction mapping, where $\alpha(x,y)=1$ for all $x,y\in X$ and $\psi(t)=kt$ for all $t\geq 0$ and some $k\in[0,1)$.

Theorem 1.5. [4] Let (X,d) be a complete metric space and $T:X\to X$ be an (α,ψ)-contraction mapping. Suppose that

(i) T is α-admissible;

(ii) there exists $x_0\in X$ such that $(x_0,Tx_0)\geq$; and

(iii) T is continuous.

Then there exists $u\in X$ such that $Tu=u$.

In 1977, Jaggi [3] introduced a new concept namely 'rational type contraction mappings' and proved the existence of fixed points of such mappings.

Theorem 1.6. [3] Let T be a continuous self-map defined on a complete metric space (X,d). Suppose that T satisfies the following condition: There exist $\alpha,\beta\in[0,1)$ with $\alpha+\beta<1$ such that

$$d(Tx,Ty)\leq\alpha\frac{d(x,Tx)d(y,Ty)}{d(x,y)}+\beta d(x,y)\text{ for all } x,y\in X,\ x\neq y. \quad (1.6.1)$$

Then T has a fixed point in X.

Here we note that a mapping $T: X \to X$, X a metric space that satisfies (1.6.1) is called a *Jaggi contraction map* on X.

Definition 1.7. [5]Let (X,d) be a metric space and $T: X \to X$ be a selfmap of X.

If there exist two functions $\alpha: X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$\alpha(x,y)d(Tx,Ty) \le \psi(M(x,y))$ for all $x, y \in X$, where

$$M(x,y) = \max\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\},$$

thenwe say that T is *a generalized* (α,ψ)*-contraction mapping*.

Theorem 1.8. [5]Let (X,d) be a complete metric space and $T: X \to X$ be a generalized (α,ψ)-contraction mapping. Suppose that

(i) T is α-admissible;

(ii) there exists $x_0 \in$ X such that $\alpha(x_0, Tx_0) \ge 1$; and

(iii) T is continuous.

Then there exists $u \in X$ such that $Tu = u$.

Harjani, Lopez and Sadarangani [2] extended Theorem 1.6 to the context of partially ordered complete metric space.

Theorem 1.9. [2] Let $(X,\preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let $T: X \to X$ be a non-decreasing mapping such that

$$d(Tx,Ty) \le \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y) \qquad (1.9.1)$$

for all $x, y \in X$ with $x \succeq y$, $x \ne y$ where $0 \le \alpha, \beta < 1$ with $\alpha + \beta < 1$.

Also, assume either

(i) T is continuous; (or)

(ii) If a non-decreasing sequence $\{x_n\}$ in X is such that $x_n \to x$ as $n \to \infty$ then $x = \sup\{x_n\}$.

If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

A map T that satisfies the inequality (1.9.1) is called Jaggi contraction map in partially ordered metric spaces.

In 2013, Arshad, Karapinar and Ahmad [1] extended Theorem 1.6 to almost Jaggi contraction type mappings in partially ordered metric spaces.

Definition 1.10. [1] Let $(X,d,\preceq)$ be a partially ordered metric space. A selfmapping T on X is called an *almost Jaggi contraction* if it satisfies the following condition: There exist $\alpha,\beta\in[0,1)$ with $\alpha+\beta<1$ and $L\geq 0$ such that,

$$\begin{aligned} d(Tx,Ty) \leq \alpha\frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y) \\ + L\min\{d(x,Tx),d(x,Ty),d(y,Tx)\} \end{aligned} \tag{1.10.1}$$

for any distinct $x,y\in X$ with $x\preceq y$.

Theorem 1.11. [1] Let $(X,d,\preceq)$ be a complete partially ordered metric space. Suppose that a selfmap $T:X\to X$ is a continuous and non-decreasing mapping that satisfies the following inequality : there exist $\alpha,\beta\in[0,1)$ with $\alpha+\beta<1$ and $L\geq 0$ such that

$$d(Tx,Ty) \leq \alpha\frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y) + L\min\{d(x,Ty),d(y,Tx)\} \tag{1.11.1}$$

for all $x,y\in X$ with $x\neq y$ and $x\preceq y$. Suppose there exists $x_0\in X$ with $x_0\preceq Tx_0$. Then T has a unique fixed point.

Remark 1.12: Since every almost Jaggi contraction satisfies the inequality (1.11.1), it follows that the conclusion of Theorem 1.11 is valid under the replacement of condition (1.11.1) by almost Jaggi contraction in Theorem 1.11.

In the following, we introduce almost generalized (α,ψ)-contractions with rational expressions in complete partially ordered metric spaces.

Definition 1.13. Let $(X,\preceq)$ be a partially ordered metric space and suppose that $T:X\to X$ be a mapping. If there exist two functions $\alpha: X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(\mathrm{M}(x,y)) + L.\mathrm{N}(x,y)\text{, where} \tag{1.13.1}$$

$$\mathrm{M}(x,y)=\begin{cases}\max\{d(x,y),\dfrac{d(x,Tx)d(y,Ty)}{d(x,y)},\dfrac{d(x,Ty)d(y,Tx)}{d(x,y)},\dfrac{d(y,Ty)d(x,Ty)}{d(x,y)},\\ \qquad\qquad\dfrac{d(x,Tx)d(x,Ty)}{d(x,y)}\} & if\ x\preceq y, x\neq y\\ 0\ if\ x=y\end{cases}$$

and $\mathrm{N}(x,y)=\min\{d(x,Tx),d(x,Ty),d(y,Tx)\}$, $x,y\in X$ with $x\preceq y$, then we say that T is an *almost generalized* (α,ψ)*-contraction map with rational expressions*.

Note: Clearly, a map T that satisfies (1.9.1) with $\alpha+\beta<1$ also satisfies the inequality (1.13.1) with $\alpha(x,y)=1$ for all $x,y\in X$, $L=0$ and $\psi(t)=(\alpha+\beta)t, t\geq 0$ so that T is an almost generalized (α,ψ)-contraction map with rational expressions. But, the following example suggests that its converse need not be true.

Example 1.14. Let $X=\{0,1,2,3,4\}$ with the usual metric. We define a partial order $\preceq$ on X as follows, $\preceq:=\{$(0,0), (1,1), (2, 2), (3,3), (4, 4), (0, 2), (0,3), (0, 4), (1, 2), (1,3), (1, 4), (2,3), (2, 4), (3, 4)$\}$. Let $A=\{$(0,0), (1,1), (2, 2), (3,3), (4, 4), (0,1), (1,0), (0, 2), (2,0), (2,3), (3, 4), (4,3)$\}$ and $B=\{$(3,2),(1,2),(2,1),(0,4),(4,0),(0,3),(3,0),(1,3),(3,1),(1,4),(4,1),(2,4),(4,2)$\}$. We define $T:X\to X$ by T0 = T1 = 0,T2 = 3 and T3 = T4 = 4. We define $\alpha:X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}\frac{3}{2} & if\ (x,y)\in A\\ 0 & if\ (x,y)\in B\end{cases}$$ and $\psi:[0,\infty)\to[0,\infty)$ by $\psi(t)=\frac{5}{6}t$.

Case (i): $x=2$ and $y=0$.

In this case, $d(T2,T0)=3$, $\mathrm{M}(2,0)=3$ and $\mathrm{N}(2,0)=1$.

$$\begin{aligned}\alpha(x,y)d(Tx,Ty)&=\alpha(2,0)d(T2,T0)\\ &=\frac{9}{2}\leq\psi(3)+L.1=\psi(\mathrm{M}(2,0))+L.\mathrm{N}(2,0)\\ &=\psi(\mathrm{M}(x,y))+L.\mathrm{N}(x,y)\end{aligned}$$

holds with $L=3$.

Case (ii): $x=2$ and $y=3$.

In this case, $d(T2,T3)=1$, $\mathrm{M}(2,3)=2$ and $\mathrm{N}(2,3)=0$.

$$\begin{aligned}\alpha(x,y)d(Tx,Ty) &= \alpha(2,3)d(T2,T3)\\ &= \frac{3}{2} \le \psi(2)+L.0=\psi(\mathrm{M}(2,3))+L.\mathrm{N}(2,3)\\ &= \psi(\mathrm{M}(x,y))+L.\mathrm{N}(x,y)\end{aligned}$$

holds for any $L \ge 0$. If $x,y \in B$ then the inequality (1.13.1) holds trivially. Hence, from case (i) and case (ii), we choose $L=3$, so that T is *an almost generalized* (α,ψ) *-contraction map with rational expressions* with L = 3.

Also we observe that the inequality (1.10.1) fails to hold.

For, by choosing $x=2$ and $y=3$ we have

$$\begin{aligned}d(T2,T3)=1 &\nleq \alpha(1)+\beta(1)+L.0<1\\ &= \alpha\frac{d(2,T2)d(3,T3)}{d(2,3)}+\beta d(2,3)+L.\min\{d(2,T3),d(3,T2)\}\end{aligned}$$

i.e., T is not an *almost Jaggi contraction map.*

Further, we observe that the inequality (1.9.1) also fails to hold.

For, when $(x,y)=(2,0)$ we have

$$\begin{aligned}d(T2,T0)=3 \nleq \alpha(0)+\beta(2) &\le \alpha\frac{d(2,T2)d(0,T0)}{d(2,0)}+\beta d(2,0)\\ &= \alpha\frac{d(x,Tx)d(y,Ty)}{d(x,y)}+\beta d(x,y)\end{aligned}.$$

This shows that the inequality (1.9.1) fails to hold so that T is not a *Jaggicontration* map.

Thus we conclude that the class of almost generalized (α,ψ)-contraction maps with rational expressions is more general than the class of almost Jaggi contraction maps and the class of all Jaggi contraction maps also.

In Section 2, we prove the existence of fixed points of almost generalized (α,ψ)-contraction maps with rational expressions. Further, we obtain the uniqueness of fixed point under an additional assumption ‘Condition (H)’. In Section 3, we provide examples.

2. MAIN RESULTS

Theorem 2.1. Let $(X,\preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let $T: X \to X$ be a non-decreasing mapping. Suppose that there exist two functions $\alpha: X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that T is an almost generalized (α,ψ) -contraction map with rational expressions. Also, assume that

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ with $x_0 \preceq Tx_0$; either

(iii) T is continuous (or)

(iv) if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ as $n \to \infty$ then $x = \sup\{x_n\}$; and also $\alpha(x_0,x) \geq 1$ and $\alpha(x,Tx) \geq 1$

Then T has a fixed point in X.

Proof: By (ii), we have $x_0 \in X$ be such that $\alpha(x_0,Tx_0) \geq 1$.

We define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for $n = 0,1,2\ldots$. (2.1.1)

If $x_{n+1} = x_n$ for some n, then x_n is a fixed point of T.

Hence w. l. g. we assume that $x_{n+1} \neq x_n$ for all n.

We have $\alpha(x_0,x_1) = \alpha(x_0,Tx_0) \geq 1$ and since T is α-admissible, we have

$$\alpha(Tx_0,Tx_1) = \alpha(x_1,x_2) \geq 1. \qquad (2.1.2)$$

On continuing this process, we have

$$\alpha(x_n,x_{n+1}) \geq 1 \text{ for all } n \geq 0. \qquad (2.1.3)$$

Since T is non-decreasing and $x_0 \preceq Tx_0 = x_1$, we have $x_1 = Tx_0 \preceq Tx_1 = x_2$.

On continuing this process, we have $x_n \preceq x_{n+1}$ for all $n \geq 0$. (2.1.4)

By using (1.13.1), (2.1.3) and (2.1.4), we have

$$
\begin{aligned}
d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\
&\le \alpha(x_n,x_{n-1})d(Tx_n,Tx_{n-1}) \\
&\le \psi(\max\{\frac{d(x_n,Tx_n)d(x_{n-1},Tx_{n-1})}{d(x_n,x_{n-1})},\frac{d(x_n,Tx_{n-1})d(x_{n-1},Tx_n)}{d(x_n,x_{n-1})} \\
&\qquad ,\frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_{n-1})}{d(x_n,x_{n-1})}, \\
&\qquad d(x_n,x_{n-1}),\frac{d(x_n,Tx_n)d(x_n,Tx_{n-1})}{d(x_n,x_{n-1})}\}) \\
&\quad +L\min\{d(x_n,Tx_n),d(x_n,Tx_{n-1}),d(x_{n-1},Tx_n)\} \\
&\le \psi(\max\{d(x_n,x_{n-1}),\frac{d(x_n,x_{n+1})d(x_{n-1},x_n)}{d(x_n,x_{n-1})}, \\
&\quad \frac{d(x_n,x_n)d(x_{n-1},x_{n+1})}{d(x_n,x_{n-1})},\frac{d(x_{n-1},x_n)d(x_n,x_n)}{d(x_n,x_{n-1})}, \\
&\qquad \frac{d(x_n,x_{n+1})d(x_n,x_n)}{d(x_n,x_{n-1})}\}) \\
&\quad + L\min\{d(x_n,x_{n+1}),d(x_n,x_n),d(x_{n-1},x_{n+1})\} \\
&= \psi(\max\{d(x_n,x_{n-1}),\ d(x_n,x_{n+1})\}.
\end{aligned} \tag{2.1.5}
$$

Now, if $\max\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$ then we have

$d(x_n,x_{n+1}) \le \psi(d(x_n,x_{n+1})) < d(x_n,x_{n+1})$,

a contradiction.

Hence, from (2.1.5) we have,

$\max\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\} = d(x_n,x_{n-1})$ so that

$d(x_n,x_{n+1}) \le \psi(d(x_n,x_{n-1}))$ for all n ≥ 1. Hence by induction, it follows that

$$d(x_n,x_{n+1}) \le \psi^n(d(x_1,x_0)). \tag{2.1.6}$$

From (2.1.6) and using triangular inequality, for all k ≥ 1, we have

$$
\begin{aligned}
d(x_n,x_{n+k}) &\le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + ... + d(x_{n+k-1},x_{n+k}) \\
&= \sum_{p=n}^{n+k-1} d(x_p,x_{p+1}) \\
&\le \sum_{p=n}^{+\infty} \psi^n(d(x_1,x_0)) \to 0 \quad \text{as n} \to \infty.
\end{aligned}
$$

This implies that $\{x_n\}$ is a Cauchy sequence in X.

Since (X,d) is complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{2.1.7}$$

First we assume that T is continuous. In this case, from (2.1.1), we obtain that

$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz$.

Hence z is a fixed point of T.

Now, suppose that the condition (iv) holds. Since $\{x_n\}$ is a non-decreasing sequence and $x_n \to x$ we have $x = \sup\{x_n\}$.

Particularly $x_n \preceq x$ for all n. Since T is non-decreasing, we have $Tx_n \preceq Tx$ for all n.

i.e., $x_{n+1} \preceq Tx$ for all n.

Moreover, $x_n \preceq x_{n+1} \preceq Tx$ for all n and $x = \sup\{x_n\}$, we get $x \preceq Tx$.

Let us now consider the sequence $\{y_n\}$ that is constructed as follows:

$y_0 = x, y_{n+1} = Ty_n$, $n = 0,1,2...$.

Then $y_0 \preceq Ty_0$ and by condition (iv), we have $\alpha(x_0,x) \geq 1$ and $\alpha(x,x) \geq 1$. i.e., $\alpha(x_0,y_0) \geq 1$ and $\alpha(y_0,Ty_0) \geq 1$. Since T is non-decreasing, we obtain that $\{y_n\}$ is a non-decreasing sequence and $\{y_n\}$ is cauchy (similar to the argument to show $\{x_n\}$ is cauchy) $y_n \to y$ (say), $y \in X$. Again, by the first part of the condition (iv), we have $y = \sup\{y_n\}$. Since $x_n \preceq x = y_0 \preceq Tx = Ty_0 \preceq y_n \preceq y$ for all n. Now $\alpha(x_0,y_0) \geq 1$ implies $\alpha(Tx_0,Ty_0) = \alpha(x_1,y_1) \geq 1$,

$\alpha(Tx_1,Ty_1) = \alpha(x_2,y_2) \geq 1$.

On continuing this process, we have $\alpha(x_{n+1},y_{n+1}) \geq 1$, for $n = 0,1,2...$.

Suppose that $x \neq y$. Now from (1.13.1), we have

$$\begin{aligned}
d(x_{n+1}, y_{n+1}) &= d(Tx_n, Ty_n) \\
&\leq \alpha(x_n, y_n) d(Tx_n, Ty_n) \\
&\leq \psi(\max\{d(x_n, y_n), \frac{d(x_n, Tx_n) d(y_n, Ty_n)}{d(x_n, y_n)}, \frac{d(x_n, Ty_n) d(y_n, Tx_n)}{d(x_n, y_n)}, \\
&\quad \frac{d(y_n, Ty_n) d(x_n, Ty_n)}{d(x_n, y_n)}, \frac{d(x_n, Tx_n) d(x_n, Ty_n)}{d(x_n, y_n)}\}) \\
&\quad + L \min\{d(x_n, Tx_n), d(x_n, Ty_n), d(y_n, Tx_n)\} \\
&\leq \psi(\max\{d(x_n, y_n), \frac{d(x_n, x_{n+1}) d(y_n, y_{n+1})}{d(x_n, y_n)}, \frac{d(x_n, y_{n+1}) d(y_n, x_{n+1})}{d(x_n, y_n)}, \\
&\quad \frac{d(y_n, y_{n+1}) d(x_n, y_{n+1})}{d(x_n, y_n)}, \\
&\quad \frac{d(x_n, x_{n+1}) d(x_n, y_{n+1})}{d(x_n, y_n)}\}) \\
&\quad + L \min\{d(x_n, x_{n+1}), d(x_n, y_{n+1}), d(y_n, x_{n+1})\}.
\end{aligned}$$

On letting $n \to \infty$ we have

$$\begin{aligned}
d(x, y) &\leq \psi(\max\{d(x, y), \frac{d(x, x) d(y, y)}{d(x, y)}, \frac{d(x, y) d(y, x)}{d(x, y)}, \frac{d(y, y) d(x, y)}{d(x, y)}, \\
&\quad \frac{d(x, x) d(x, y)}{d(x, y)}\}) + L \min\{d(x, x), d(x, y), d(y, x)\} \\
&= \psi(\max\{d(x, y), 0, d(x, y), 0, 0\}) + L.0 \\
&= \psi(d(x, y)) < d(x, y),
\end{aligned}$$

a contradiction.

Hence $x = y$, and we have $x \preceq Tx = y_0 \preceq y_n \preceq y = x$.

Therefore x is a fixed point of T.

Corollary 2.2. Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping. Suppose that there exists a function $\alpha : X \times X \to [0, \infty)$ and constant $k \in [0,1)$ and $L \geq 0$ such that

$\alpha(x, y) d(Tx, Ty) \leq k\mathrm{M}(x, y) + L.\mathrm{N}(x, y)$ (2.2.1) for all $x, y \in X$ with $x \preceq y$, $x \neq y$. Also, assume that

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$; either

(iii) T is continuous (or)

(iv) $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ as $n \to \infty$ then $x = \sup\{x_n\}$; and also $\alpha(x_0, x) \geq 1$ and $\alpha(x, Tx) \geq 1$.

Then T has a fixed point in X.

Proof: The conclusion of this corollary follows by taking $\psi(t) = kt, t \geq 0$ in Theorem 2.1.

Remark 2.3. (i) Theorem 1.9 follows as a corollary to Corollary 2.2, since the inequality (1.9.1) implies the inequality (2.2.1) with $k = \alpha + \beta < 1$; $\alpha(x, y) = 1$ for all $x, y \in X$ and $L = 0$. Hence Theorem 1.9 is a corollary to Theorem 2.1.

(ii) Theorem 1.11 follows as a corollary to Corollary 2.2, since the inequality (1.11.1) implies the inequality (2.2.1) with $k = \alpha + \beta < 1$; and $\alpha(x, y) = 1$ for all $x, y \in X$.

Now we prove the uniqueness of fixed point of T under 'condition (H)' and it is the following:

Condition (H): For all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Theorem 2.4. Let $(X, \preceq)$ be partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$\alpha(x, y)d(Tx, Ty) \leq \psi(\mathrm{M}(x, y)) + L.\mathrm{N}(x, y)$, where (2.4.1)

$$\mathrm{M}(x,y) = \begin{cases} \max\{d(x,y), \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)}, \dfrac{d(x,Ty)d(y,Tx)}{d(x,y)}, \\ \qquad \dfrac{d(x,Tx)d(x,Ty)}{d(x,y)}\} \ \ if\ x \preceq y, x \neq y \\ \qquad 0\ if\ x = y \end{cases}$$

and $N(x,y)=\min\{d(x,Tx),d(x,Ty),d(y,Tx)\}$, $x,y\in X$ with $x\preceq y$.

Also, assume that

(i) T is α-admissible;

(ii) there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$ with $x_0\preceq Tx_0$; either

(iii) T is continuous (or)

(iv) if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n\to x$ as $n\to\infty$ then $x=\sup\{x_n\}$; and also $\alpha(x_0,x)\geq 1$ and $\alpha(x,Tx)\geq 1$.

If condition (H) holds, then T has a unique fixed point.

Proof: Since the inequality (2.4.1) implies (1.13.1), it follows that T is a (α,ψ)-contraction map, and hence by Theorem 2.1, T has a fixed point. Suppose that $x,y\in X$ are two fixed points of T. By condition (H), there exists $z\in X$ such that

$x\preceq z$ and $y\preceq z$, $\alpha(x,z)\geq 1$ and $\alpha(y,z)\geq 1$.

Put $z=z_0$ and choose $z_1\in X$ such that $z_1=Tz_0$.

We define a sequence $\{z_n\}$ in X by $z_{n+1}=Tz_n$ for all $n\geq 0$. Then $x\preceq z_0$ and $y\preceq z_0$, $\alpha(x,z_0)\geq 1$ and $\alpha(y,z_0)\geq 1$. By using the non-decreasing property of T, we have $Tx\preceq Tz_0$ and $y\preceq Tz_0$. Hence $x\preceq z_1$ and $y\preceq z_1$.

On continuing this process, we have

$x\preceq z_n$ and $y\preceq z_n$ for $n\geq 0$ (2.4.2)

Now, since T is α-admissible, we have

$\alpha(Tx,Tz_0)\geq 1$ and $\alpha(Ty,Tz_0)\geq 1$. Hence $\alpha(x,z_1)\geq 1$ and $\alpha(y,z_1)\geq 1$.

On repeating this process, we have

$\alpha(x,z_n)\geq 1$ and $\alpha(y,z_n)\geq 1$ for $n\geq 0$. (2.4.3)

In (2.4.2), if $x=z_n$ for some n, then $Tx=Tz_n$ so that $x=z_{n+1}$. Also, we have $x=z_m$ for $m\geq n$ so that $\lim_{n\to\infty} z_n=x$.

Hence w. l. g we assume that $x \neq z_n$ for all n.

By using (2.4.1) with (2.4.3) we have

$$\begin{aligned}
d(x,z_{n+1}) &= d(Tx,Tz_n) \\
&\leq \alpha(x,z_n)d(Tx,Tz_n) \\
&\leq \psi(\max\{d(x,z_n),\frac{d(x,Tx)d(z_n,Tz_n)}{d(x,z_n)},\frac{d(x,Tz_n)d(z_n,Tx)}{d(x,z_n)}, \\
&\qquad \frac{d(x,Tx)d(x,Tz_n)}{d(x,z_n)}\}) + L\min\{d(x,Tx),d(x,Tz_n),d(z_n,Tx)\} \\
&\leq \psi(\max\{d(x,z_n),\frac{d(x,x)d(z_n,z_{n+1})}{d(x,z_n)},\frac{d(x,z_{n+1})d(z_n,x)}{d(x,z_n)}, \\
&\qquad \frac{d(x,x)d(x,z_{n+1})}{d(x,z_n)}\}) + L\min\{d(x,Tx),d(x,z_{n+1}),d(z_n,x)\} \\
&\leq \psi(\max\{d(x,z_n),d(x,z_{n+1})\} \qquad (2.4.4)
\end{aligned}$$

Now, if $\max\{d(x,z_n),d(x,z_{n+1})\} = d(x,z_{n+1})$ then we have

$$d(x,z_{n+1}) \leq \psi(d(x,z_{n+1})) < d(x,z_{n+1}),$$

a contradiction.

Hence, from (2.4.4), we have

$\max\{d(x,z_{n+1}),d(x,z_n)\} = d(x,z_n)$ so that

$$\begin{aligned}
d(x,z_{n+1}) &\leq \psi(d(x,z_n)) = \psi(\psi(d(x,z_{n-1}))) \\
&\leq \psi^2(d(x,z_{n-1})) \leq \psi^3(d(x,z_{n-2})) \leq .... \leq \psi^n(d(x,z_1)) \to 0 \text{ as n} \to \infty.
\end{aligned}$$

Therefore $\lim_{n\to\infty} z_n = x$. (2.4.5)

By applying the similar argument to $\{y_n\}$ it follows that

$$\lim_{n\to\infty} z_n = y. \qquad (2.4.6)$$

From (2.4.5) and (2.4.6) we have $x = y$.

This completes the proof of the Theorem.

In the following, we provide examples in support of the results obtained in Section 2.

Example3.1. Let $X=[0,4]$ with the usual metric. We define a partial order $\preceq$ on X by $\preceq := \{(x,y): x,y\in[0,2), x=y\}\cup\{(x,y): x,y\in[2,4], x\preceq y\}$. Then $(X,\preceq)$ is a partially ordered set.

We define $T: X\to X$ by $T(x)=\begin{cases}\frac{x}{2} & if\ 0\le x<1\\ \frac{3x}{2}-1 & if\ 1\le x<\frac{10}{3}\\ 4 & if\ \frac{10}{3}\le x\le 4,\end{cases}$

and $\alpha: X\times X\to[0,\infty)$ by $\alpha(x,y)=\begin{cases}1 & if\ \ 2\le x\le 4\ and\ \ y=4\\ 0 & otherwise.\end{cases}$

Here we note that T is non-decreasing on X and continuous on X. Moreover, we choose $x_0=\frac{10}{3}\in X$, then $\alpha(x_0,Tx_0)=\alpha(\frac{10}{3},4)\ge 1$ and $\frac{10}{3}\le T\frac{10}{3}=4$, for $x_0=\frac{10}{3}, x_1=Tx_0=4$ and $x_n=Tx_{n-1}=4$ for all $n\ge 1$ and hence $x=\lim_{n\to\infty}x_n=4$. Also $\alpha(x_0,x)=\alpha(\frac{10}{3},4)\ge 1$ and $\alpha(x,Tx)=\alpha(4,T4)\ge 1$.

Now, we show that T is α-admissible.

Case (i) $2\le x<\frac{10}{3}$ and $y=4$.

In this case, $Tx\in[2,4)$ and $Ty=T4=4$.

Therefore, by the definition of α we have $\alpha(Tx,T4)=1$.

Case (ii) $\frac{10}{3}<x\le 4$ and $y=4$.

In this case, $Tx=4$ and $Ty=T4=4$ and hence, $\alpha(Tx,Ty)=\alpha(4,4)=1$. Therefore, T is α-admissible. Now, we verify the inequality (1.13.1) by choosing $\psi\in\Psi$ given by $\psi(t)=\frac{t}{2}$ for $t\ge 0$ and $L=\frac{1}{2}$.

Case (i) $2 \le x < \frac{10}{3}$ and $y = 4$.

In this case, $\alpha(x,y)=1, Tx=\frac{3x}{2}-1, Ty=4$ and $\alpha(x,y)d(Tx,Ty)=5-\frac{3}{2}x$,

$\mathrm{M}(x,y)=\max\{4-x,0,5-\frac{3}{2}x,0,\frac{x}{2}-1\}=4-x$ and

$\mathrm{N}(x,y)=\min\{4-x,5-\frac{3}{2}x,\frac{x}{2}-1\}=\{\begin{matrix}\frac{x}{2}-1 & \text{if } 2\le x\le 3\\ 5-\frac{3}{2}x & \text{if } 3\le x\le\frac{10}{3}.\end{matrix}$

Sub Case (i):

$$5-\frac{3}{2}x=\alpha(x,y)d(Tx,Ty)\le\frac{1}{2}(4-x)+\frac{1}{2}(\frac{x}{2}-1)=\psi(\mathrm{M}(x,y))+L.\mathrm{N}(x,y).$$

Sub Case (ii):

$$5-\frac{3}{2}x=\alpha(x,y)d(Tx,Ty)\le\frac{1}{2}(4-x)+\frac{1}{2}(5-\frac{3}{2}x)=\psi(\mathrm{M}(x,y))+L.\mathrm{N}(x,y).$$

Case (ii) $\frac{10}{3}\le x\le 4$ and $y=4$.

In this case, $d(Tx,Ty)=d(4,4)=0$, hence we have

$$\alpha(x,y)d(Tx,Ty)=0\le\psi(\mathrm{M}(x,y))+L.\mathrm{N}(x,y).$$

Therefore T satisfies the inequality (1.13.1) and hence T satisfies all the hypotheses of Theorem 2.1 and T has three fixed points 0, 2 and 4.

Here we note that if $L=0$ in the inequality (1.13.1), then for $x=2$ and $y=4$ we have $\alpha(2,4)d(T2,T4)=2\not\le\psi(\mathrm{M}(2,4))=\psi(2)$ for any $\psi\in\Psi$. Hence the inequality (1.13.1) fails to hold when $L=0$. This example shows the importance of L in the inequality (1.13.1) of Theorem 2.1.

Further, we observe that the inequality (1.9.1) also fails to hold. For, by choosing $(x,y)=(2,4)$ we have

$$d(T2,T4)=2\not\le\alpha.0+\beta.2<1=\alpha\frac{d(2,T2)d(4,T4)}{d(2,4)}+\beta d(2,4)$$

for any $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$.

Hence Theorem 1.9 isnot applicable. Therefore, by Remark 2.3 (i) it follows that Theorem 2.1 is a generalization of Theorem 1.9.

Remark 3.2. We note that T also satisfies the inequality (2.4.1) with the same α and ψ that are mentioned in Example 3.1. But, for $x = \frac{1}{4}$ and $y = \frac{1}{2}$, and for any z in X, $\frac{1}{4} \not\preceq z$ and $\frac{1}{2} \not\preceq z$; also $\alpha(\frac{1}{4}, z) \not\geq 1$, $\alpha(\frac{1}{2}, z) \not\geq 1$. Hence condition (H) of Theorem 2.4 fails to hold and T, α and ψ satisfy all the remaining hypotheses of Theorem 2.4. We observe that T has more than one fixed point namely 0, 2 and 4.

The following is an example in support of Theorem 2.1 when (iv) of Theorem 2.1 holds, but T fails to be continuous.

Example 3.3. Let $X = [0,2]$ with the usual metric. We define a partial order $\preceq$ on X by $\preceq := \{(x,y) : x, y \in [0,2), x = y\} \cup \{(0,2),(1,2),(\frac{3}{2},2)\}$.

Let $A = \{(x,y) : x, y \in [0,2), x = y\} \cup \{(0,2),(1,2),(\frac{3}{2},2)\}$

and $B = \{(x,y) \in X \times X : x \neq y, x \neq 0, 1, \frac{3}{2} \text{ and } y \neq 2\}$.

We define $T : X \to X$ by $T(x) = \begin{cases} 1 - x & \text{if } 0 \leq x < 1 \\ 2 & \text{if } 1 < x \leq 2 \end{cases}$ and $\alpha : X \times X \to [0, \infty)$ by $\alpha(x,y) = \begin{cases} \frac{3}{2} & \text{if } (x,y) \in A \\ 0 & \text{if } (x,y) \in B. \end{cases}$

Here we note that T is non-decreasing on X, not continuous and α-admissible. Moreover, we choose $x_0 = \frac{3}{2} \in X$, then $\alpha(x_0, Tx_0) = \alpha(\frac{3}{2}, 2) \geq 1$ and $\frac{3}{2} \preceq T\frac{3}{2}$, for $x_0 = \frac{3}{2}, x_1 = Tx_0 = 2$ and $x_n = Tx_{n-1} = 2$ for all $n \geq 1$ and hence $x = \lim_{n\to\infty} x_n = 2$. Also $\alpha(x_0, x) = \alpha(\frac{3}{2}, 2) \geq 1$ and $\alpha(x, Tx) = \alpha(2, T2) \geq 1$.

Now, we verify the inequality (1.13.1) by choosing $\psi \in \Psi$ given by $\psi(t) = \frac{2}{5}t$ for $t \geq 0$ and $L = 3$.

Case (i): $x = 0$ and $y = 2$.

In this case, $\alpha(x, y) = \frac{3}{2}, Tx = 1, Ty = 2$ and $\alpha(x, y)d(Tx, Ty) = \frac{3}{2}$,

$\mathrm{M}(x, y) = \max\{2,0,1,0,1\} = 2$ and $\mathrm{N}(x, y) = \min\{2,1,1\} = 1$.

Hence, we have

$$\frac{3}{2} = \alpha(x, y)d(Tx, Ty) \leq \frac{2}{5}(2) + 3.1 = \psi(\mathrm{M}(x, y)) + L.\mathrm{N}(x, y).$$

Case (ii): $x = 1$ and $y = 2$.

In this case, $\alpha(x, y) = \frac{3}{2}$, $Tx = 0, Ty = 2$ and $\alpha(x, y)d(Tx, Ty) = 3$,

$\mathrm{M}(x, y) = \max\{1,0,2,0,1\} = 2$ and $\mathrm{N}(x, y) = \min\{1,1,2\} = 1$.

Hence, we have

$$3 = \alpha(x, y)d(Tx, Ty) \leq \frac{2}{5}(2) + 3.1 = \psi(\mathrm{M}(x, y)) + L.\mathrm{N}(x, y).$$

Case (iii): $x = \frac{3}{2}$ and $y = 2$.

In this case, the inequality (1.13.1) trivially hold.

From all the cases considered above, T satisfies the inequality (1.13.1) and hence T satisfies all the hypotheses of the Theorem 2.1 and T has two fixed points $\frac{1}{2}$ and 2.

Here we note that if $L = 0$ in the inequality (1.13.1), then for $x = 1$ and $y = 2$ we have $\alpha(1,2)d(T1, T2) = 3 \nleq \psi(2) = \psi(\mathrm{M}(1,2))$ for any $\psi \in \Psi$ so that the inequality (1.13.1) fails to hold when $L = 0$, which shows the importance of L in Theorem 2.1.

Further, we observe that the inequality (1.9.1) fails to hold. For, by choosing $(x,y)=(1,2)$ we have

$$d(T1,T2)=2\not\leq\alpha.0+\beta.1<1=\alpha\frac{d(1,T1)d(2,T2)}{d(1,2)}+\beta d(1,2).$$

Hence Theorem 1.9 is not applicable. Therefore, by Remark 2.3 it follows that Theorem 2.1 is a generalization of Theorem 1.9.

One more example in support of Theorem 2.1 is the following:

Example 3.4. Let X,T,Ψ,α and partial order $\preceq$ be as in Example 1.14. Then T is α-admissible and choose $x_0=2\in X$. Then $\alpha(x_0,Tx_0)=\alpha(2,3)\geq 1$ and $2\preceq T2$. Also T is an almost generalized (α,ψ)-contraction map with rational expressions with $L=3$ and is verified in Example 1.14. Hence T satisfies all the hypotheses of Theorem 2.1 and T has two fixed points 0 and 4.

Here we note that the inequality (1.13.1) fails to hold when $L=0$. For, when $x=2$ and $y=0$ we have $\alpha(2,0)d(T2,T0)=\frac{9}{2}\not\leq\psi(3)=\psi(M(2,0))$ for any $\psi\in\Psi$, which shows the importance of L in Theorem 2.1.

Further this T is neither Jaggi contraction nor almost Jaggi contraction and it is observed in Example1.14.

Hence, by Remark 2.3 (i) and (ii), we conclude that Theorem 2.1 is a generalization of Theorem 1.9 and Theorem 1.11.

We conclude this paper with the following example in support of Theorem 2.4.

Example 3.5. Let $X=\{0,1,2\}$ with the usual metric. We define a partial order $\preceq$ on X by $\preceq:=\{(0,0),(1,1),(2,2),(0,1),(0,2),(1,2)\}$. Let $A=\{(0,0),(1,1),(2,2),(0,2),(2,0),(1,2)\}$ and $B=\{(0,1),(1,0),(2,1)\}$. We define $T:X\to X$ by $T0=2,T1=0$ and $T2=2$. We define $\alpha:X\times X\to[0,\infty)$ by $\alpha(x,y)=\begin{cases}\frac{3}{2} & if\ (x,y)\in A\\ 0 & if\ (x,y)\in B.\end{cases}$

Then T is continuous, non-decreasing and α-admissible. We choose $x_0=0\in X$. Clearly $x_0\preceq Tx_0$ and $\alpha(x_0,Tx_0)=\alpha(2,2)=\frac{3}{2}\geq 1$. Further, T

satisfies the inequality (2.4.1) by choosing $\psi \in \Psi$ given by $\psi(t) = \frac{4}{5}t$ for $t \geq 0$ and $L = 2$. Hence T satisfies all the hypotheses of Theorem 2.4 and 2 is the unique fixed point of T.

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