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Zeros of Lacunary Type of Polynomials

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Abstract In this paper we use matrix methods and Gereshgorian disk Theorem to present some interesting generalizations of some well-known results concerning the distribution of the zeros of polynomial. Our results include as a special case some results due to A .Aziz and a result of Simon Reich-Lossar.

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1. INTRODUCTION AND STATEMENT OF RESULTS

The following result due to Cauchy [4] is well known in the theory of the distribution of the zeros of a polynomial.

Theorem A. Let

$$P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

be a polynomial of degree *n* then all the zeros of P(z) lie in the disk

$$|z| < 1 + A. \qquad (1)$$

where $A = \max |a_j|, j = 0,1,2,\ldots,n-1$.

About forty years ago, in connection with Cauchy's Classical result (Theorem A) Simon Reich proposed and among others Lossers [6] verified that if $a_{n-1} = 0$, Q>1, then all the zeros of

$$P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \text{,}$$

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lie in the circle

$$|z| \leq Q + Q^2 + \cdots + Q^{n-1} \tag{2}$$

Aziz [2] generalized the problem to lacunary polynomials and showed that the assertion (2), remains valid even if we do not assume that Q>1. In fact he proved:

Theorem B. Let

$$P(z) = a_n z^n + a_r z^r + \ldots + a_1 z + a_0,$$

$a_r \neq 0, 0 < r \leq n - 1$ be a polynomial of degree n $\geq$ 2, with real or complex coefficients if

$$Q = \left\{ \underset{0 \leq j \leq r}{Max} \left| \frac{a_j}{a_n} \right| \right\}^{1/n}$$

then all the zeros of P(z) lie in the disk

$$|\, z \,| \leq Q + Q^2 + \ldots + Q^{r+1} \tag{3}$$

Where 0 $\leq$ r $\leq$ n $-$1. Other results of similar type were obtained among others by Alzer [1], Bell [3], Guggenheimer [5]. Mohammad [7], Rahman [8], Walsh [10] (see also [9]).

As a generalization of Theorem B, we prove:

Theorem 1. Let

$$P(z) = a_n z^n + a_r z^r + \ldots + a_1 z + a_0$$

$a_r \neq 0$ 0 $\leq$ r $\leq$ n $-$ 1 be a polynomial of degree n $\geq$ 2, with real or complex coefficients if t is any given positive number and

$$Q_t = \left\{ Max_{0 \leq j \leq r} \left| \frac{a_j}{a_n} \right| t^{n-1} \right\}^{\frac{1}{n}} \tag{4}$$

then all the zeros of P(z) lie in the disk

$$|z| \leq \frac{1}{t}\left\{Q_t + Q_t^2 + \ldots + Q_t^{r+1}\right\} \tag{5}$$

where $0 \leq r \leq n-1$.

Taking $t = 1$, in equation (5), this reduces to Theorem B.

We next present the following result which provides an interesting refinement of Theorem 1.

Theorem 2. Let

$$P(z) = a_n z^n + a_r z^r + \ldots + a_1 z + a_0$$

$a_r \neq 0 \leq r \leq n-1$ be a polynomial of degree $n \geq 2$, with real or complex coefficients if t is any given positive number and

$$Q_t = \left\{Max_{0 \leq j \leq r}\left|\frac{a_j}{a_n}\right| t^{n-1}\right\}^{\frac{1}{n}},$$

then all the zeros of P(z) lie in the disk

$$|z| \leq \frac{1}{t}\left\{Q_t + Max(Q_t^2, Q_t^{r+1})\right\} \tag{6}$$

where $1 \leq r \leq n-1$. The following result immediately follows from Theorem 2 by taking $t = 1$:

Corollary 1. Let

$$P(z) = a_n z^n + a_r z^r + \ldots + a_1 z + a_0$$

$a_r \neq 0 \leq r \leq n-1$ be a polynomial of degree $n \geq 2$, with real or complex coefficients if t is any given positive number and

$$Q_t = \left\{Max_{0 \leq j \leq r}\left|\frac{a_j}{a_n}\right| t^{n-1}\right\}^{\frac{1}{n}},$$

then all the zeros of P(z) lie in the disk

$$|z| \leq Q + Max\left\{Q^2 + ... + Q^{r+1}\right\} \quad (7)$$

where $1 \leq r \leq n-1$,

PROOF OF THE THEOREMS

Proof of Theorem 1. The companion matrix of the polynomial

$$P(z) = a_n z^n + a_r z^r + \ldots + a_1 z + a_0$$

$a_r \neq 0$ $0 \leq r \leq n-1$ of degree n is

$$C = \begin{pmatrix} 0 & 0 & .... & 0 & .... & 0 & \frac{-a_0 t^{n-1}}{a_n Q_t^{n-1}} \\ \frac{Q_t}{t} & 0 & .... & 0 & .... & 0 & \frac{-a_1 t^{n-2}}{a_n Q_t^{n-1}} \\ ... & ... & .... & .... & .... & ..... & \\ 0 & 0 & .... & \frac{Q_t}{t} & ..... & 0 & \frac{-a_r t^{n-r-1}}{a_n Q_t^{n-r-1}} \\ .... & .... & .... & .... & .... & .... & .... \\ 0 & 0 & .... & 0 & .... & \frac{Q_t}{t} & 0 \end{pmatrix}$$

By hypothesis,

$$Q_t = \left\{ Max_{0 \leq j \leq r} \left| \frac{a_j}{a_n} \right| t^{n-j} \right\}^{\frac{1}{n}}$$

therefore,

$$\left| \frac{a_j}{a_n} \right| t^{n-j} \leq Q_t^n \quad for \ \ j = 0,1,2,...,r. \ and \ Q_t \neq 0. \quad (7)$$

We take the matrix

$$\mathrm{P} = \mathrm{diag}\left\{ \left(\frac{Q_t}{t} \right)^{n-1}, \left(\frac{Q_t}{t} \right)^{n-2}, ..., \left(\frac{Q_t}{t} \right), 1 \right\}$$

and form the matrix

$$P^{-1}CP \begin{pmatrix} 0 & 0 & \ldots & 0 & \ldots & 0 & \frac{-a_0 t^{n-1}}{a_n Q_t^{n-1}} \\ \frac{Q_t}{t} & 0 & \ldots & 0 & \ldots & 0 & \frac{-a_1 t^{n-2}}{a_n Q_t^{n-1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots & \frac{Q_t}{t} & \ldots & 0 & \frac{-a_r t^{n-r-1}}{a_n Q_t^{n-r-1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & \ldots & \frac{Q_t}{t} & 0 \end{pmatrix}$$

Applying Gereshgorian Theorem to the columns of P^{-1} CP and noting (7), it follows that all the eigen values of the matrix P^{-1} CP lie in the circle

$$\begin{aligned} |z| &\leq Max\left\{\frac{Q_t}{t}, \sum_{j=0}^{r}\left|\frac{a_j}{a_n}\right| \frac{t^{n-j-1}}{Q_t^{n-j-1}}\right\} \\ &\leq \frac{1}{t} Max\left\{Q_t, \sum_{j=0}^{r} Q_t^{j+1}\right\} \\ &= \frac{1}{t}\left\{Q_t + Q_t^2 + \ldots + Q_t^{r+1}\right\} \end{aligned}$$

Since the matrix P^{-1} CP is similar to the matrix C and the eigen values of C are the zeros of the polynomial P(z), it follows that all the zeros of P(z) lie in the circle

$$|z| \leq \frac{1}{t}\left\{Q_t + Q_t^2 + \ldots + Q_t^{r+1}\right\}$$

Which completes the proof of Theorem 1.

Proof of Theorem 2. The companion matrix of the polynomial

$$P(z) = a_n z^n + a_r z^r + \ldots + a_1 z + a_0$$

$a_r \neq 0$ $0 \leq r \leq n-1$ of degree n is given by

$$C=\begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & \frac{-a_0 t^{n-1}}{a_n Q_t^{n-1}} \\ \frac{Q_t}{t} & 0 & \cdots & 0 & \cdots & 0 & \frac{-a_1 t^{n-2}}{a_n Q_t^{n-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{Q_t}{t} & \cdots & 0 & \frac{-a_r t^{n-r-1}}{a_n Q_t^{n-r-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & 0 & & \frac{Q_t}{} & 0 \end{pmatrix}$$

Proceeding similarly as in the proof of Theorem 1 and noting that

$$P=\operatorname{diag}\left\{\left(\frac{Q_t}{t}\right)^{n-1},\left(\frac{Q_t}{t}\right)^{n-2},\ldots,\left(\frac{Q_t}{t}\right),1\right\}$$

$$Q_t=\left\{Max_{0\le j\le r}\left|\frac{a_j}{a_n}\right|t^{n-j}\right\}^{\frac{1}{n}}$$

It follows that the matrix

$$P^{-1}CP\quad\begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & \frac{-a_0 t^{n-1}}{a_n Q_t^{n-1}} \\ \frac{Q_t}{t} & 0 & \cdots & 0 & \cdots & 0 & \frac{-a_1 t^{n-2}}{a_n Q_t^{n-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{Q_t}{t} & \cdots & 0 & \frac{-a_r t^{n-r-1}}{a_n Q_t^{n-r-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & \frac{Q_t}{t} & 0 \end{pmatrix}$$

Applying Gereshgorian Theorem to the columns of P^{-1} CP and noting (7), it follows that all the eigen values of the matrix P^{-1} CP therefore that of C lie in the circle

$$
\begin{aligned}
|z| &\leq Max_{1\leq j\leq r}\left\{\left|\frac{a_0}{a_n}\right|\frac{t^{n-1}}{Q_t^{n-1}},\frac{Q_t}{t}+\left|\frac{a_j}{a_n}\right|\frac{t^{n-j-1}}{Q_t^{n-j-1}}\right\} \\
&\leq \frac{1}{t}Max_{1\leq j\leq r}\left\{Q_t, Q_t+Q_t^{j+1}\right\} \\
&= \frac{1}{t}\left\{Q_t+Max(Q_t^2, Q_t^{r+1}\right\}
\end{aligned}
$$

Since the matrix P^{-1} CP is similar to the matrix C and the eigen values of C are the zeros of the polynomial P(z), therefore we conclude that all the zeros of P(z) lie in the circle denoted by (4). This proves Theorem 2 completely.

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