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# On $\chi_s$-Orthogonal Matrices

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**Abstract:** In this paper we, introduced the concept of $\chi_s$-orthogonal matrices and extended some results of Abara et al, [3] in the context of secondary transpose.

**Key words:** $\chi_s$-orthogonal matrices, s-unitary matrices, s-normal matrices.
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## 1. INTRODUCTION AND PRELIMINARIES

The concept of secondary transpose and related matrices was initiated by [1, 2]. An $n \times n$ matrix A is said to be s-symmetric if $A^s = A$; an A is said to be s-skew symmetric if $A^s = -A$; an A is s-normal if $AA^s = A^s A$; an $\chi_s$-orthogonal if $A^s = A^{-1}$; and an A is said to be s-unitary if $A^s = A^{-1}$ [4, 5]. Here we introduce the matrix, namely $\chi_s$-orthogonal and derived some results related to $\chi_s$-orthogonal matrices. An $n \times n$ non-singular matrix A is said to $\chi_s$-orthogonal, if $\chi_s(A) = A^{-1}$, where $\chi_s(A) = S^{-1}A^sS$ and S satisfies the condition $S^2 = \pm I$; an A is said to be $\chi_s$-symmetric if $\chi_s(A) = A$; and an A is called $\chi_s$-skew symmetric if $\chi_s(A) = -A$.

**Remark 1.1.** *Let A be an $n \times n$ matrix and it is said to be $\chi_s$-orthogonal, if one of the following conditions must hold*

*(i)* $S^{-1}A^sS = A^{-1}$
*(ii)* $A^sSA = S$
*(iii)* $A^sS = SA^{-1}$
*(iv)* $A^s = SA^{-1}S^{-1} = S(SA)^{-1}$

## 2. $\chi_s$-ORTHOGONAL MATRICES

**Theorem 2.1.** *Let A be an $n \times n$ matrix and it is $\chi_s$-symmetric ($\chi_s$-skew symmetric), then so are (a). $A^{-1}$ (b). $-A$ and (c). $\lambda A$ ($\lambda$ is an arbitrary constant).*

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*Proof.* A is $\chi_s$-symmetric, if $\chi_s(A) = A \Rightarrow S^{-1} A^s S = A$.

(a) $S^{-1}(A^{-1})^s S = S^{-1}(A^s)^{-1} S$
$= S^{-1}(SAS^{-1})^{-1} S$
$= S^{-1}[(SA)S^{-1}]^{-1} S$
$= S^{-1}SA^{-1}S^{-1}S$
$= A^{-1}$

(b) $S^{-1}(-A)^s S = S^{-1}(-A^s)S$
$= -S^{-1}A^s S$
$= -A$

(c) $S^{-1}(\lambda A)^s S = S^{-1}\lambda A^s S$
$= \lambda S^{-1}A^s S$
$= \lambda S^{-1}SAS^{-1}S$
$= \lambda A$

Similarly we have to prove the same for $\chi_s$-skew symmetric.

**Corollary 2.2.** *Let $S \in S_n$ and $S^2 = I$. If A is an $n \times n$ matrix and it is $\chi_s$-symmetric ($\chi_s$-skew symmetric), then $A^s$ is also.*

*Proof. A is $\chi_s$-symmetric, if $\chi_s(A) = A \Rightarrow A^s SA^{-1} = S$.*

$$\begin{aligned}(A^s)^s S(A^s)^{-1} &= AS(A^s)^{-1}\\ &= S^{-1}A^s SS(A^s)^{-1}\\ &= S^{-1}A^s(A^s)^{-1}\\ &= S^{-1}\\ &= S\end{aligned}$$

Similarly we have to prove the same for $\chi_s$-skew symmetric.

**Remark 2.3.** *If A is an $n \times n$ matrix and it is $\chi_s$-symmetric, then $(A+A^s)$ and $A^sA$ are not.*

**Theorem 2.4.** *If A and B are $\chi_s$-symmetric ($\chi_s$-skew symmetric) with same size, then (a). $A + B$ and (b). $A - B$ are also.*

*Proof.* A is $\chi_s$-symmetric, if $\chi_s(A) = A \Rightarrow S^{-1}A^s S = A$.

(a) $S^{-1}(A + B)^s S = S^{-1}(A^s + B^s)S$
$= (S^{-1}A^s + S^{-1}B^s)S$
$= S^{-1} A^s S + S^{-1}B^s S$
$= (A + B)$

(b) $S^{-1}(A - B)^s S = S^{-1}(A^s - B^s)S$
$= (S^{-1} A^s - S^{-1}B^s)S$
$= S^{-1}A^s S - S^{-1}B^s S$
$= (A - B)$

Similarly we have to prove the same for $\chi_s$-skew symmetric.

**Theorem 2.5.** *Let A be an n × n matrix and it is $\chi_s$-orthogonal, then (a). –A and (b). $A^{-1}$ are, also $\chi_s$-orthogonal.*

*Proof.* A is $\chi_s$-orthogonal, if $A^sSA = S$.

(a) $(-A)^sS(-A) = A^sSA$
$= SA^{-1}S^{-1}SA$
$= SA^{-1}A$
$= S$

Therefore $-A$ is $\chi_s$-orthogonal

(b) $(A^{-1})^sSA^{-1} = (As)^{-1} SA^{-1}$
$= (SA^{-1}S^{-1})^{-1}SS^{-1}A^sS$
$= ((SA^{-1})S^{-1})^{-1}A^sS$
$= (S^{-1})^{-1}(SA^{-1})^{-1}A^sS$
$= S(A^{-1})^{-1}SA^sS$
$= SASA^sS$
$= SASSA^{-1}S^{-1}S$
$= SAA^{-1}$
$= S$

Therefore $A^{-1}$ is $\chi_s$-orthogonal.

**Corollary 2.6.** *Let $S \in S_n$ and $S^2 = I$. If A is an n × n matrix and it is $\chi_s$-orthogonal, then (a). $A^s$ and (b). $AA^s$ are also.*

*Proof. A* is $\chi_s$*-orthogonal, if $A^sSA = S$.*

(a) $(A^s)^sS(A^s) = AS(A^s)$
$= ASSA^{-1}S^{-1}$
$= AA^{-1}S^{-1}$
$= S^{-1}$
$= S$

(b) $(AA^s)^sS(AA^s) = (A^s)^sA^sSAA^s$
$= AA^sSAA^s$
$= ASA^{-1}S^{-1}SASA^{-1}S^{-1}$
$= ASA^{-1}ASA^{-1}S^{-1}$
$= ASSA^{-1}S^{-1}$
$= AA^{-1}S^{-1}$
$= S^{-1}$
$= S$

**Theorem 2.7.** *Product of two $\chi_s$–orthogonal matrix is also $\chi_s$-orthogonal.*

*Proof.* Let A and B are $\chi_s$-orthogonal. Then by the definition $A^sSA = S$ and $B^sSB = S$.

$$\begin{aligned} \Rightarrow (AB)^s S(AB) &= B^s A^s SAB \\ &= SB^{-1}S^{-1}SA^{-1}S^{-1}SAB \\ &= SB^{-1}A^{-1}AB \\ &= S(AB)^{-1}AB \\ &= S \end{aligned}$$

Therefore AB is $\chi_s$-orthogonal.

**Lemma 2.8.** *Let $S \in S_n$ and A* is *an $n \times n$ s-normal, $\chi_s$-orthogonal with $-1 \in \sigma(A)$. Then there exist $\chi_s$–orthogonal $P, Q \in M_n(C)$ such that P is positive definite, Q is s-unitary P and Q commute, and $A = PQ$.*

**Lemma 2.9,** *Let $S \in S_n$ and A is an $n \times n$ s-normal, $\chi_s$-orthogonal with $-1 \in \sigma(A)$.*

*(a) Then there exist s-hermitian $\chi_s$-skew symmetric $P_1, Q_1 \in M_n(C)$ such that $P_1$ and $Q_1$ commute and $A = e^{P_1 + iQ_1}$.*

*(b) A is positive definite if and only if there exists a s-hermitian $\chi_s$-skew symmetric $P \in M_n(C)$ such that $A = e^P$.*

*(c) If A* is *s-unitary, then there exists a s-hermitian $\chi_s$-skew symmetric $Q \in M_n(C)$ such that $A = e^{iQ}$.*

**Corollary 2.10.** *Let $S \in S_n$ and A is an $n \times n$, $\chi_s$-orthogonal matrix. Then A is s-hermitian and positive definite if and only if there exists a s-hermitian $\chi_s$-skew symmetric $P \in M_n$ such that $A = e^P$.*

**Corollary 2.11.** *Let Q be a $n \times n$, s-hermitian $\chi_s$-skew symmetric matrix, then $A\bar{A}^s = e^Q$.*

**Theorem 2.12.** *Let $S \in S_n$ and A is an $n \times n$, $\chi_s$-orthogonal matrix. Then there exist $\chi_s$-orthogonal $P, V \in M_n$, with* P *be positive definite and* V is *s-unitary such that $A = PV$.*

**Theorem 2.13.** *Let $S \in S_n$ be given. Let $Q \in M_n$ be s-orthogonal, and set $U \equiv QSQ^s$ and $V \equiv QAQ^s$. Then*

*(a) An $n \times n$ matrix A is $\chi_s$-orthogonal $\Leftrightarrow V^sUV = U$.*

*(b) An $n \times n$ matrix A is $\chi_s$-symmetric $\Leftrightarrow V^sUV^{-1} = U$.*

*(c) An $n \times n$ matrix A is $\chi_s$-skew symmetric $\Leftrightarrow V^sUV^{-1} = -U$.*

*Proof.*

(a) $V^s U V = U \Leftrightarrow (QAQ^s)^s QSQ^s (QAQ^s) = QSQ^s$

$$\Leftrightarrow [(QA)Q^s]^s QSQ^s QAQ^s = QSQ^s$$
$$\Leftrightarrow (Q^s)y(QA)^s QSAQ^s = QSQ^s$$
$$\Leftrightarrow QA^s Q^s QSAQ^s = QSQ^s$$
$$\Leftrightarrow QA^s SAQ^s = QSQ^s$$
$$\Leftrightarrow A^s SA = S$$

*(b)* $V^s U V^{-1} = U \Leftrightarrow [QAQ^s]^s QSQ^s [QAQ^s]^{-1} = QSQ^s$

$$\Leftrightarrow [(QA)Q^s]^s QSQ^s [(QA)Q^s)^{-1} = QSQ^s$$
$$\Leftrightarrow (Q^s)^s (QA)^s QSQ^s [(Q^s)^{-1} (QA)^{-1} = QSQ^s$$
$$\Leftrightarrow QA^s Q^s QSQ^s [(Q^s)^{-1} A^{-1} Q^{-1}] = QSQ^s$$
$$\Leftrightarrow QA^s SQ^s (Q^s)^{-1} A^{-1} Q^{-1} = QSQ^s$$
$$\Leftrightarrow QA^s SA^{-1} Q^{-1} = QSQ^{-1}$$
$$\Leftrightarrow A^s SA^{-1} = S$$

(c) $V^s U V^{-1} = -U \Leftrightarrow [QAQ^s]^s QSQ^s [QAQ^s]^{-1} = -QSQ^s$

$$\Leftrightarrow [(QA)Q^s)^s]^s QSQ^s [(QA)Q^s]^{-1} = -QSQ^s$$
$$\Leftrightarrow (Q^s)^s (QA)^s QSQ^s [(Q^s)^{-1} (QA)^{-1}] = -QSQ^s$$
$$\Leftrightarrow QA^s Q^s QSQ^s [(Q^s)^{-1} A^{-1} Q^{-1}] = -QSQ^s$$
$$\Leftrightarrow QA^s SQ^s (Q^s)^{-1} A^{-1} Q^{-1} = -QSQ^s$$
$$\Leftrightarrow QA^s SA^{-1} Q^{-1} = -QSQ^{-1}$$
$$\Leftrightarrow A^s SA^{-1} = -S$$

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