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Extension of Blasius Newtonian Boundary Layer to Blasius Non-Newtonian Boundary Layer

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ABSTRACT

Blasius equation is very well known and it aries in many boundary layer problems of fluid dynamics. In this present article, the Blasius boundary layer is extended by transforming the stress strain term from Newtonian to non-Newtonian. The extension of Blasius boundary layer is discussed using some non-newtonian fluid models like, Power-law model, Sisko model and Prandtl model. The Generalised governing partial differential equations for Blasius boundary layer for all above three models are transformed into the non-linear ordinary differewntial equations using the one parameter deductive group theory technique. The obtained similarity solutions are then solved numerically. The graphical presentation is also explained for the same. It concludes that velocity increases more rapidly when fluid index is moving from shear thickninhg to shear thininhg fluid.

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Nomenclature:

u ,v - Velocity components in X, Y directions respectively
U - Main stream velocities in X direction
n- flow behaviour indices
ψ - stream function
G -group notation
a -group parameter
a^0 -identity element of a group
$M(a)$ - real constant
η - similarity variable
f - similarity functions
α_i, β_i - real constants/parameters
A, B, C – fluid parameters

1. Introduction

If the Reynolds number is large, the inertial forces will be predominant and in such a case the effect of viscosity can be considered to be confined in a thin layer known as a velocity boundary layer; adjacent to a solid boundary. Any disturbance created in the laminar flow in the boundary layer is ultimately dumped. This is known as the Laminar Boundary layer. The boundary layer concept was first introduced by Ludwig Prandtl, a German aerodynamicist in 1904.

In the case of fluid motions for which the measured pressure distribution nearly agrees with the perfect fluid theory, such as the flow past the streamlined body or the airfoil, the influence of viscosity at high Reynolds numbers is confined to a very thin layer in the immediate neighborhood of the solid wall. If the condition of no-slip were not to be satisfied in the case of a real fluid, there would be no appreciable difference between the field of the flow of the real fluid as compared with that of a perfect fluid. The fact that at the wall the fluid adheres to it means, however, that frictional forces retard the motion of the fluid in a thin layer near the wall. In that thin layer, the velocity of the fluid increases from zero as the wall (no-slip) to its full value which corresponds to external frictionless flow. The layer under consideration is called the boundary layer.

In 1904, Ludwig Prandtl (1904) has analysed that, for the equation of the boundary layer flow, almost half of the terms of Navier-Stokes equations are negligible. This reduced Navier-Stokes equations are called Prandtl boundary layer equations. Lateron, Blasius (1908) proposed a similarity

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solution for the case in which the free stream velocity is constant, which corresponds to the boundary layer over a flat plate that is oriented parallel to the free flow. Which is known as Blasius boundary layer and similarity solution (third order non-linear ODE) is called Blasius equation.

Solution of Blasius equation is discussed earlier by many researchers (Wang 2004), Hashim (2006), Robin (2013). Liu et al. (2011) has solved the Blasius equation using the Variational Iteration method. They had derived the approximate analytical solution and also compared the results with that available in the literature. The Numerical solutions of the Blasius Equation with Crocco-Wang transformation was obtained by Asaithambi (2016). The direct second order finite difference solution is given by him. Ganji et al. (2009) had discussed the solution of Blasius equation applying the Homotopy Perturbation method. The obtained results are then compared with that available in literature. Chaotic behavior in the flow along a wedge modeled by the Blasius equation along with the numerical solutions was discussed by Basu et al. (2011). Another class of boundary layer problem for a stretching sheet relevant to the Blasius equation was studied by Sakiadis (1961).

All the above research was done for the Newtonian case. Probably the first time, the generalized Blasius equations was discussed by Benlahsen et al. (2008). They considered the non-linear relationship for the strees and rate of deformation, the non-Newtonian case, in the Blasius boundary layer. The well-known Poewer – Law fluid model is used for the strees and rate of deformation term. The generalized Blasius equations was derived for the same.

In the Present paper, the extension of Blasius boundary layer is discussed using some non-newtonian fluid models like, Power-law model, Sisko model and Prandtl model. All the fluid models are explained systematically by Patel et al. (2020). The Generalised governing partial differential equations for Blasius boundary layer for all above three models are transformed into the non-linear ordinary differential equations using the one parameter deductive group theory technique. The obtained similarity solutions are then solved numerically. The graphical presentation is also explained for the same. The new approach discussed in the present work to Blasius boundary layer will be useful to develop boundary layer equations for other fluid problems also. It can be applied to some other real world problems.

2. Governing Equations

The steady-state two dimensional generalised Blasius boundary layer flow over a semi-infinite plate is considered. The incompressible flow of non-Newtonian fluid with constant density and viscosity is assumed.

2.1. Blasius Boundary Layer with Power-Law Fluid Model

The boundary layer differential equations for the flow along a thin flat plate was first discussed by Prandtl (1904). Lateron Blasius (1908) has considered free flow by taking constant free stream velocity in the Prandtl boundary layer equations. That boundary layer was discussed by Blasius therefore it is called Blasius boundary layer, In which the fluid was considered Newtonian. By taking the non-linear stress-strain relationship, in the Blasius boundary layer equation, it is extanded/generalized to Blasius boundary layer equations for non-Newtonian fluid flows. The flow geometry is shown in the Figure 1. The governing Partial differential equations equations 1-4, (following Blasius (1908)) for steady-state two dimensional generalised Blasius boundary layer flow of Power-law fluid over a semi-infinite plate are; (Schlichting (2000))

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=n\left(\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

The boundary conditions are

$$u=v=0 \quad at \qquad y=0, \tag{3}$$

$$u=U(x) \quad at \qquad y\to\infty. \tag{4}$$

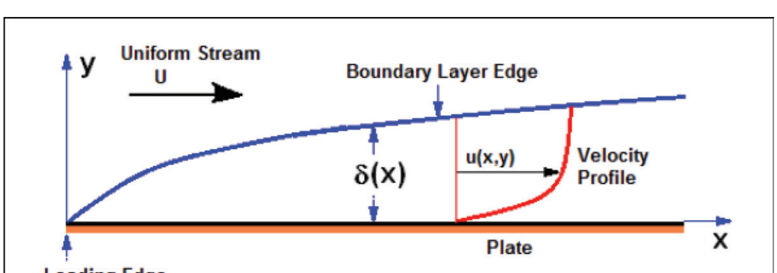

Figure 1: Blasius flow of a boundary layer on a flat plate.

Introducing the stream function $\psi(x,y)$, where $u=\dfrac{\partial\psi}{\partial y}$ & $v=-\dfrac{\partial\psi}{\partial x}$, the continuity equation (1) is satisfied therefore it need not to solve. And using the same stream function in the momentum equation given by equation (2) and the boundary conditions given by equations (3) and (4), we obtaine the below equations (5), (6) and (7) respectively.

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=n\left(\frac{\partial^2\psi}{\partial y^2}\right)^{n-1}\frac{\partial^3\psi}{\partial y^3}, \tag{5}$$

The boundary conditions are

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial y}=0 \quad at \quad y=0, \tag{6}$$

$$\frac{\partial\psi}{\partial y}=U_0=1 \quad at \quad y\to\infty. \tag{7}$$

Now, to reduce the above equations (5-7) with two independent variables in the equations with one independent variable, we apply the one-parameter deductive group theory technique (Moran et al. 1967, 1968, 1968, 1968).

Introducing the group G, (Patel, Patel and Timol 2017)

$$G:\begin{cases}\tilde{x}=h^x(a)\,x+k^x(a),\\ \tilde{y}=h^y(a)\,y+k^y(a),\\ \tilde{\psi}=h^\psi(a)\,\psi+k^\psi(a),\\ \tilde{U}=h^U(a)U+k^U(a).\end{cases} \tag{8}$$

Equation (5) is said to be transformed invariantly for some function M(a), whenever ,

$$\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}}-\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}-n\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)^{n-1}\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3} =M_1(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-n\left(\frac{\partial^2\psi}{\partial y^2}\right)^{n-1}\frac{\partial^3\psi}{\partial y^3}\right]. \tag{9}$$

Therefore, from the above equation (9), the following relations (10) are obtained,

$$\frac{h^{2\psi}}{h^x h^{2y}}=\frac{h^{2\psi}}{h^x h^{2y}}=\left(\frac{h^\psi}{h^{2y}}\right)^{n-1}\frac{h^\psi}{h^{3y}}=M_1(a), \tag{10}$$

Also from the invariance of the auxiliary conditions

$$k^y=0,\ h^\psi=h^y,h^U=1,k^U=0. \tag{11}$$

Using (11) in (8), the group G is

$$G:\begin{cases}\tilde{x}=h^{3y}x+k^x,\\ \tilde{y}=h^y y,\\ \tilde{\psi}=h^{2y}\psi+k^\psi,\\ \tilde{U}=U\end{cases} \tag{12}$$

$$\sum_{i=1}^{4}(\alpha_i S_i+\beta_i)\frac{\partial g}{\partial S_i}=0;\quad S_i=x,y,\psi,U. \tag{13}$$

$$\Rightarrow(\alpha_1x+\beta_1)\frac{\partial g}{\partial x}+(\alpha_2y+\beta_2)\frac{\partial g}{\partial y} +(\alpha_3\psi+\beta_3)\frac{\partial g}{\partial\psi}+(\alpha_4U+\beta_4)\frac{\partial g}{\partial U}=0, \tag{14}$$

Here $\alpha_i=\dfrac{\partial h^{S_i}}{\partial a}$ & $\beta_i=\dfrac{\partial k^{S_i}}{\partial a}$; $i=1,2,3,4.$ (15)

$$\begin{aligned}\alpha_1&=\frac{\partial h^x}{\partial a}=\frac{\partial h^{3y}}{\partial a}=3h^{2y}\frac{\partial h^y}{\partial a}=3\frac{\partial h^y}{\partial a}=3\alpha_2 \quad ,(\because h^y\ is\,identity\,at\ a_0)\\ \alpha_2&=\frac{\partial h^y}{\partial a},\\ \alpha_3&=\frac{\partial h^\psi}{\partial a}=\frac{\partial h^{2y}}{\partial a}=2h^y\frac{\partial h^y}{\partial a}=2\frac{\partial h^y}{\partial a}=2\alpha_2 \quad ,(\because h^y\ is\,identity\,at\ a_0)\\ \alpha_4&=\frac{\partial h^U}{\partial a}=0,(\because h^y\ is\,identity\,at\ a_0)\end{aligned} \tag{16}$$

$$\beta_1=\frac{\partial k^x}{\partial a},\ \beta_2=\frac{\partial k^y}{\partial a}=0,\ \beta_3=\frac{\partial k^\psi}{\partial a},\ \beta_4=\frac{\partial k^U}{\partial a}=0. \tag{17}$$

Now, the characteristic equation from the equation (14) is:

$$\frac{dx}{(\alpha_1x+\beta_1)}=\frac{dy}{(\alpha_2y+\beta_2)}=\frac{d\psi}{(\alpha_3\psi+\beta_3)}=\frac{dU}{(\alpha_4U+\beta_4)}. \tag{18}$$

Solving first two relation of equation (18) for η, we have

$$\eta = y(\alpha_1 x+\beta_1)^{-\frac{1}{3}}. \qquad (19)$$

Solving first and third relation of equation (18) for $f_1(\eta)$, we have

$$\psi = (\alpha_1 x+\beta_1)^{\frac{2}{3}} f(\eta) - \frac{\beta_3}{2\alpha_1}. \qquad (20)$$

Solving first and last relation of equation (18) for U, we have

$$U = \text{Constant} = 1 \qquad (21)$$

Using the equations (19), (20) and (21) and its derivatives in equations (5)-(7), the below similarity equation (22) alongwith the boundary condition (23) are obtained.

$$n(f'')^{n-1} f''' + \frac{1}{n+1} f f'' = 0, \qquad (22)$$

$$f(0)=0,\ f'(0)=0, f'(1)=1. \qquad (23)$$

The graph of the velocity versus eta is shown by Figure 2.

2.2. Blasius Boundary Layer with Sisko Fluid Model

The governing Partial differential equations for steady-state two dimensional generalised Blasius boundary layer flow of Sisko fluid over a semi-infinite plate are ;

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad (24)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left\{\left[A+B\left(\frac{\partial u}{\partial y}\right)^{n-1}\right]\frac{\partial u}{\partial y}\right\}, \qquad (25)$$

The boundary conditions are

$$\begin{aligned} u=v=0 \quad &at \quad y=0,\\ u=U(x) \quad &at \quad y\to\infty. \end{aligned} \qquad (26)$$

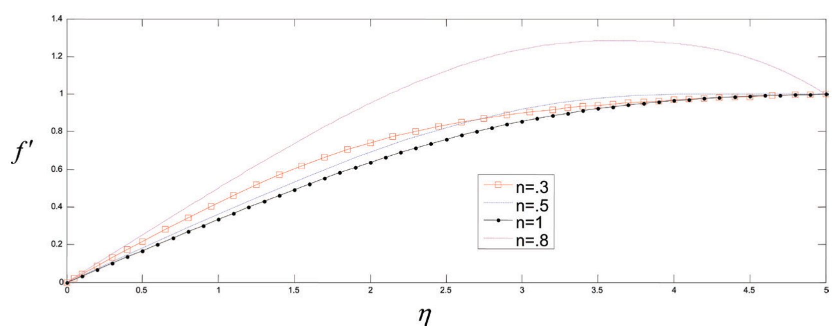

Figure 2: Velosity variation vs eta for Blasius Power-Law boundary layer.

Now, introducing the stream function and then applying the same technique, as discussed in the above case of Power-Law model, the following similarity equation (27) alongwith the boundary condition (28) are obtained.

$$a f''' + nb(f'')^{n-1} f''' + \frac{1}{n+1} f f'' = 0, \qquad (27)$$

Where, $a = \dfrac{A(\alpha_1 x+\beta_1)}{\alpha_1}$, $b = \dfrac{B}{\alpha_1}$,

$$f(0)=0,\ f'(0)=0, f'(1)=1. \qquad (28)$$

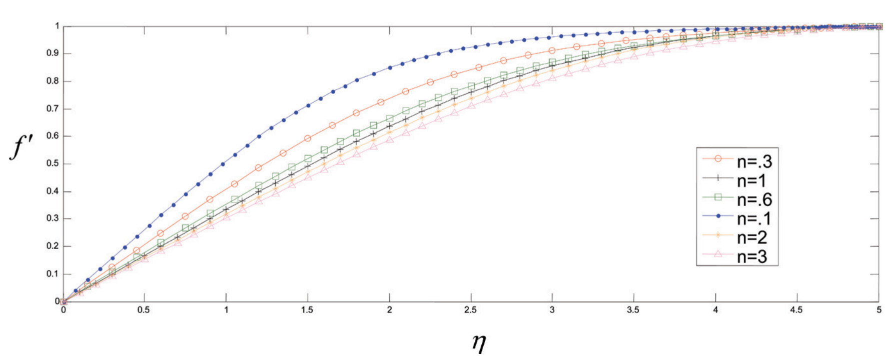

Figure 3: Velosity variation vs eta for Blasius Sisko boundary layer for a=b=0.5.

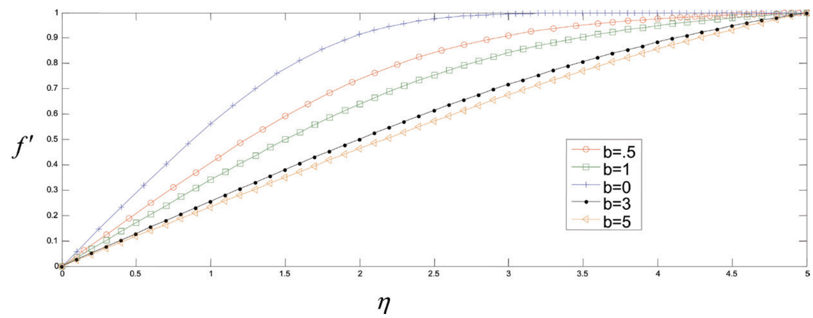

Figure 4: Velosity variation vs eta for Blasius Sisko boundary layer for n=0.3, a=0.5.

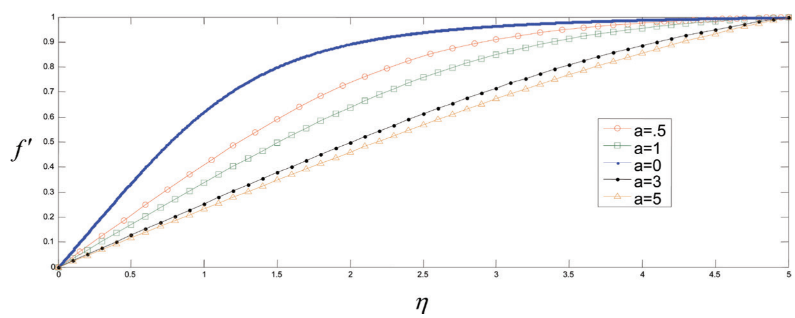

Figure 5: Velosity variation vs eta for Blasius Sisko boundary layer n=0.3, b=0.5.

2.3. Blasius Boundary Layer with Prandtl Fluid Model

The governing Partial differential equations for steady-state two dimensional generalised Blasius boundary layer flow of Power-law fluid over a semi-infinite plate are;

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{29}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left\{A\sin^{-1}\left(\frac{1}{C}\frac{\partial u}{\partial y}\right)\right\}, \tag{30}$$

The boundary conditions are

$$u=v=0 \quad at \qquad y=0, \tag{31}$$

$$u=U(x) \quad at \qquad y\to\infty. \tag{32}$$

Now, introducing the stream function and then applying the same technique, as discussed in the above case of Power-Law model, the following similarity equation (33) alongwith the boundary condition (34) are obtained.

$$a\,f'''+\frac{1}{2}a\,b(f'')^2\,f'''+\frac{1}{2}f\,f''=0, \tag{33}$$

Where, $a=\dfrac{A}{\alpha_1 C}, b=\dfrac{1}{C^2(\alpha_1 x+\beta_1)},$

$$f(0)=0,\ f'(0)=0, f'(1)=1. \tag{34}$$

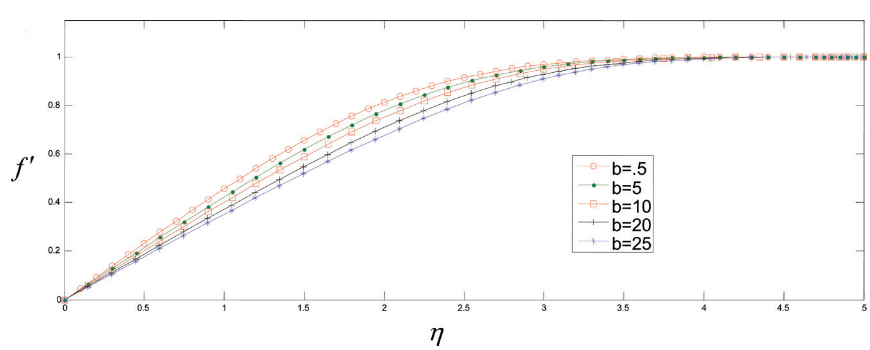

Figure 6: Velosity variation vs eta for Blasius Prandtl boundary layer a=0.5.

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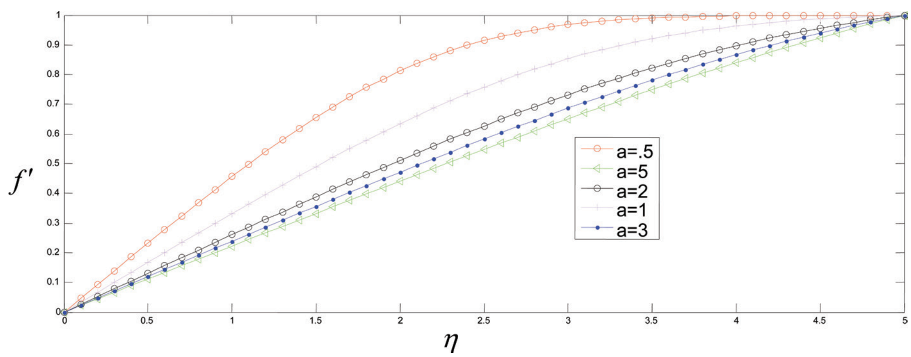

Figure 7: Velosity variation vs eta for Blasius Prandtl boundary layer b=0.5.

3. Results and Discussions

The obtained similarity equations with auxiliary conditions in each case are solved using bvp4c- MATLAB ODE solver. The variation in velocity presented graphically. The analysis of the effect of the change in fluid parameters and in fluid index for the velocity variation is very important for this type of problems. Figure 2 presents the graphs for the velocity profile for different values in fluid index ***n*** when the Power-law fluid model is considered. Velocity increases more rapidly when fluid index is moving from shear thickninhg to shear thininhg fluid. Shear thinning is the non-Newtonian behavior of fluids whose viscosity decreases under shear strain whie, a shear thickening material is one in which viscosity increases with the rate of shear strain.

Figure 3 represents the graph for the velocity profile for different values in fluid index ***n*** for fix values of fluid parameters ***a*** and ***b*** when the Sisko fluid model is considered. Here the velocity increases with the decrese in the values of *n*. Figures 4 is the graph for the velocity profile for different values of fluid parameters ***b*** with fix values of *n* and *a*. Here the velocity increases with the decrese in the values of sisko fluid parameter *b*. Figure 5 gives the graphs for the velocity profile for different values of sisko fluid parameter ***a*** and for the fix values of fluid index ***n*** and fluid parameter *b*.

Figure 6 represents the graphs for the velocity profile for different values in Prandtl fluid parameter *b* for fix value of Prandtl fluid parameter *a*. Here the velocity increases with the decrese in the values of prandtl fluid parameter *b*. Figure 7 represents the graphs for the velocity profile for different values in Prandtl fluid parameter ***a*** for fix value of Prandtl fluid parameter ***b.*** Here the velocity increases rapidly with the decrese in the values of prandtl fluid parameter ***a.***

Conclusion

Following the approach of Blasius boundary equation flow for Newtonian fluid flow, the generalized Blasius boundary layer equations for non-Newtonian fluid flow are derived. Three different non-Newtonian fluid models are considered for this extension of Blasius boundary layer as three different cases. The governing Partial differential equations of all the cases are transformed into generalized Blasius equations (similarity equations) using the one parameter Diductive group technique. The obtained generalized Blasius equations are third order non linear ordinary differential equations. The present study will help the researchers to analyse Blasius boundary layer for other non-Newtonis fluid models.

References

Asaithambi, A.: Numerical Solution of the Blasius Equation with Crocco-Wang Transformation. Journal of Applied Fluid Mechanics 9(5), 2595-2603 (2016). https://doi.org/10.18869/acadpub.jafm.68.236.25583

Basu, B., Foufoula-Georgiou, E. and Sharma, A.S.: Chaotic behavior in the flow along a wedge modeled by the Blasius equation, Nonlin. Processes Geophys 18, 171–178 (2011). https://doi.org/10.5194/npg-18-171-2011

Benlahsen, M., Guedda, M. and Kersner, R.: The Generalized Blasius equation revisited. Mathematical and Computer Modelling 47(9-10), 1063-1076 (2008). https://doi.org/10.1016/j.mcm.2007.06.019

Blasius, H.: Grenzschichten in flüssigkeiten mit kleiner reibung. Zeitschrift für Angewandte Mathematik und Physik 56, 1–37 (1908).

Ganji, D.D., Badazadeh, H., Noori, F. and Janipour, M.: An application of Homotopy perturbation method for non linear Blasius equation to boundary layer flow over a flat plate. International Journal of Nonlinear Science 7(4), 399-404 (2009).

Hashim, I.: Comments on "A new algorithm for solving classical Blasius equation" by L. Wang. Applied Mathematics and Computation 176(2), 700-703 (2006). https://doi.org/10.1016/j.amc.2005.10.016

Liu, Y. and Kurra, S.N.: Solution of Blasius Equation by Variational Iteration. Applied Mathematics 1(1), 24-27 (2011). https://doi.org/10.5923/j.am.20110101.03

Patel, M., Patel, J. and Timol, M.G.: On the solution of boundary layer flow of Prandtl fluid past a flat surface. Journal of Advanced Mathematics and Applications 6(1), 73-78 (2017). https://doi.org/10.1166/jama.2017.1128

Patel, M. and Timol, M.: Non-Newtonian fluid models and Boundary Layer flow, LAP Lambert Academic Publishing, Mauritius (2020).

Moran, M.J. and Gaggioli, R.A.: Similarity analysis if compressible boundary layer flows via group theory, Technical summary report, No.838, mathematical Research center, U.S. Army, Madison, Wisconsin (1967).

Moran, M.J. and Gaggioli, R.A.: Similarity analysis via group theory. American Institute of Aeronautics and Astronautics Journal 6(10), 2014-2016 (1968). https://doi.org/10.2514/3.4919

Moran, M.J. and Gaggioli, R.A.: A new symmetric formalism for similarity analysis, with application to boundary layer flows, Technical summary report, No.918, mathematical Research center, U.S.Army, Madison, Wisconsin (1968).

Moran, M.J. and Gaggioli, R.A.: Reduction of the number of variables in systems of partial differential equations with auxiliary conditions. SIAM Jornal on Applied Mathematics, 16(1), 202-215 (1968). https://doi.org/10.1137/0116018

Prandtl, L.: Uber Flussigkeit – shewegung bei sehr kleiner Reibung (on fluid motions with very small friction), Verhandlunger, des. III, Internationlen Mathematiker congress, Heidelburg, pp. 484-491 (1904).

Robin, W.: Some remarks on the homotopy-analysis method and series solutions to the Blasius equation. International Mathematical Forum 8(25), 1205-1213 (2013). https://doi.org/10.12988/imf.2013.3478

Sakiadis, B.C.: Boundary layer behaviour on continuous solid surfaces I. The boundary layer equations for two-dimensional and axisymmetric flows. AIChE Journal 7(1), 26–28 (1961). https://doi.org/10.1002/aic.690070108

Schlichting, H. and Gersten, K.: Boundary-Layer Theory. 8th Edition, Springer-Verlag (2000). https://doi.org/10.1007/978-3-642-85829-1

Wang, L.: A new algorithm for solving classical Blasius equation. Applied Mathematics and Computation 157(1), 1-9 (2004). https://doi.org/10.1016/j.amc.2003.06.011

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