

Some Identities Involving the Generalized Lucas Numbers

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ABSTRACT

The terms of the sequence of Lucas numbers $\{L_n\}$ can be obtained by $L_0 = 2, L_1 = 1$ and the recurrence relation $L_n = L_{n-1} + L_{n-2}$; for $n \geq 2$. There are several ways in which it has been generalized. One of these ways is by preserving the initial conditions and changing the recurrence relation. Whereas one more way is to preserve the recurrence relation and alternate the initial conditions. One of the generalizations of the Lucas sequence is the class of sequences $\{G_n^{L(a,b)}\}$ generated by the recurrence relation.

$$G_n^{L(a,b)} = \begin{cases} a G_{n-1}^{L(a,b)} + G_{n-2}^{L(a,b)}; & \text{if } n \text{ is odd} \\ b G_{n-1}^{L(a,b)} + G_{n-2}^{L(a,b)}; & \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \geq 2,$$

with the initial conditions $G_0^{L(a,b)} = 2, G_1^{L(a,b)} = a$ and a, b are any positive integers. Using the technique of generating functions, we obtain the extended Binet formula for $G_n^{L(a,b)}$. In this paper we express $G_n^{L(a,b)}$ in simple explicit form and use it to derive the recursive formula for $G_n^{L(a,b)}$ to compute the approximate value of its successor and predecessor. We also establish some amusing identities for this sequence displaying the relation between $G_n^{L(a,b)}$, classical Fibonacci sequence and classical Lucas sequence.

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1. Introduction

In the theory of numbers, the *Fibonacci sequence* has been always fertile ground for the mathematicians. At the same time, the *Lucas sequence*, being its twin sequence, also has this nature. The terms of the sequence of Fibonacci numbers $\{F_n\}$ can be obtained by $F_0 = 0, F_1 = 1$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$; for $n \geq 2$. The corresponding twin sequence, the sequence of Lucas numbers $\{L_n\}$ can be obtained by $L_0 = 2, L_1 = 1$ and the recurrence relation $L_n = L_{n-1} + L_{n-2}$; $n \geq 2$. In recent years, many interesting properties of Fibonacci numbers, Lucas numbers and their generalizations have been shown by researchers and applied to almost every field of science and art. Bacani et al 2015, Falcon 2014 Gupta et al., 2012 have generalized the sequence of Fibonacci numbers in different ways and obtain many of the interesting results. Bolat et al., 2013, Kaygisiz et al., 2012 and Shah et al., 2015 defined new generalizations of Lucas sequence and gave various identities along with extended Binet

formula for the concerned new generalizations. In this paper we further generalize these sequences and introduce *generalized Lucas numbers* as follows:

Definition: For any two positive integers a and b , the generalized Lucas sequence is defined by $G_0^{L(a,b)} = 2, G_1^{L(a,b)} = a$ and

$$G_n^{L(a,b)} = \begin{cases} a G_{n-1}^{L(a,b)} + G_{n-2}^{L(a,b)}; & \text{if } n \text{ is odd} \\ b G_{n-1}^{L(a,b)} + G_{n-2}^{L(a,b)}; & \text{if } n \text{ is even} \end{cases} \quad \text{where } n \geq 2.$$

Some initial terms of this sequence are

$$2, a, ab + 2, a^2b + 3a, a^2b^2 + 4ab + 2, a^3b^2 + 5a^2b + 5a, \dots$$

Clearly that $G_n^{L(1,1)} = L_n$. In this paper, using the techniques of generating functions, we derive the extended Binet formula for this generalized Lucas numbers and develop some interesting results for them. For convenience, throughout the paper, we use G_n for $G_n^{L(a,b)}$, when a, b are fixed.

2. Explicit Formulae for $G_n^{L(a,b)}$

Throughout we assume that α, β to be the roots of the equation $x^2 - ax - ab = 0$. This gives us $\alpha = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2}$ and $\beta = \frac{ab - \sqrt{a^2b^2 + 4ab}}{2}$. The following results are easy consequences from the values of α and β :

- (1) $\alpha\beta = -ab$
- (2) $\alpha + \beta = ab$
- (3) $\alpha - \beta = \sqrt{a^2b^2 + 4ab}$
- (4) $\alpha + 1 = \frac{\alpha^2}{ab}$
- (5) $\beta + 1 = \frac{\beta^2}{ab}$
- (6) $(\alpha + 1)(\beta + 1) = 1$
- (7) $-\beta(\alpha + 1) = \alpha$
- (8) $-\alpha(\beta + 1) = \beta$

Mathematician Jacques Philippe Marie Binet derived the Binet's formula, which is used to find the n^{th} term of the Fibonacci sequence. The formula states

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right].$$

Similar explicit formulae are known for the various generalizations of Fibonacci number. In this section, we first obtain explicit Binet – type formula for G_n .

Theorem 2.1: $G_n = \frac{ab^{1-\chi(n)}}{(ab)^{\lfloor \frac{n}{2} + 1 \rfloor}} (\alpha^n + \beta^n)$; where

$$\chi(n) = \begin{cases} 0; n \text{ is even} \\ 1; n \text{ is odd} \end{cases}.$$

Proof: Let $g(x) = G_0 + G_1x + G_2x^2 + \dots$ be the generating function for the sequence $\{G_n\}$. Then we get $-axg(x) = -axG_0 - aG_1x^2 - aG_2x^3 + \dots$ and $-x^2g(x) = -x^2G_0 - G_1x^3 - G_2x^4 + \dots$. These gives

$$g(x) = \frac{1}{\beta - \alpha} \left[a(\beta - \alpha) \sum_{n=0}^{\infty} \frac{\beta^{2n+1} + \alpha^{2n+1}}{(ab)^{n+1}} x^{2n+1} + (ab)(\beta - \alpha) \sum_{n=0}^{\infty} \frac{\beta^{2n+1} + \alpha^{2n+1}}{(ab)^{n+1}} \right]$$

$$= a \sum_{n=0}^{\infty} \frac{\beta^{2n+1} + \alpha^{2n+1}}{(ab)^{n+1}} x^{2n+1} + (ab) \sum_{n=0}^{\infty} \frac{\beta^{2n+1} + \alpha^{2n+1}}{(ab)^{n+1}}$$

$$(1 - ax - x^2)g(x) = G_0 + (G_1 - aG_0)x + (G_2 - aG_1 - G_0)x^2 + \dots$$

Using the definition of G_n , we get

$$(1 - ax - bx^2)g(x) = 2 - ax + (b - a)x[xG_1 + x^3G_3 + x^5G_5 + \dots].$$

Thus

$$g(x) = (2 - ax) \left(\frac{1}{1 - ax - bx^2} \right) + ((b - a)x) \left(\frac{1}{1 - ax - bx^2} \right) \sum_{m=1}^{\infty} G_{2m-1} x^{2m-1} \quad (1)$$

Now let $g_1(x) = \sum_{m=1}^{\infty} G_{2m-1} x^{2m-1}$. Then this gives

$$\begin{aligned} (x^4 - (ab + 2)x^2 + 1)g_1(x) &= G_1x + (G_3 - (ab + 2)G_1)x^3 \\ &+ (G_5 - (ab + 2)G_3 + G_1)x^5 + \dots \\ &= ax(1 + x^2) \end{aligned}$$

$$\text{Thus } g_1(x) = (ax(1 + x^2)) \left(\frac{1}{x^4 - (ab + 2)x^2 + 1} \right).$$

$$\text{Then by (1) we get } g(x) = (2 - ax) \left(\frac{1}{1 - ax - bx^2} \right) + ((b - a)x) (ax(1 + x^2)) \left(\frac{1}{1 - ax - bx^2} \right) \left(\frac{1}{x^4 - (ab + 2)x^2 + 1} \right).$$

$$\text{Thus } g(x) = \frac{ax^3 - (ab + 2)x^2 + ax + 2}{x^4 - (ab + 2)x^2 + 1}, \text{ which is the}$$

generating function for $\{G_n\}$. Now taking the partial fractions of denominator, we get

$$g(x) = \frac{ax^3 - (ab + 2)x^2 + ax + 2}{x^4 - (ab + 2)x^2 + 1} = \frac{1}{\beta - \alpha} \left[\frac{\alpha(\alpha + 2) - a(\alpha + 2)x}{x^2 - (\alpha + 1)} - \frac{(ab + 2)(\beta + 1) - 2 - a(\beta + 2)x}{x^2 - (\beta + 1)} \right] \quad (2)$$

Now using McLaurin Expansion, we get

$$= \sum_{n=0}^{\infty} \frac{ab^{1-\chi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1}} (\beta^m + \alpha^m) x^m ; \text{ where } (m) = \begin{cases} 0; m \text{ is even} \\ 1; m \text{ is odd} \end{cases}$$

Hence, $G_n = \frac{ab^{1-\chi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1}} (\alpha^n + \beta^n)$; where

$$\chi(n) = \begin{cases} 0; n \text{ is even} \\ 1; n \text{ is odd} \end{cases}, \text{ as required.}$$

We next use this formula obtained to express G_n as an infinite series.

Corollary 2.2: $G_n = \frac{(ab)^k}{2^{2k-1}} \sum_{r=1}^{\infty} \binom{2k}{2r} \left(1 + \frac{4}{ab}\right)^r ; n = 2k$

$$= \frac{a(ab)^k}{2^{2k}} \sum_{r=1}^{\infty} \binom{2k+1}{2r} \left(1 + \frac{4}{ab}\right)^r ; n = 2k + 1.$$

$$= \frac{c(ab)^n}{2^n} \left[\begin{aligned} & 1 + \binom{n}{1} \sqrt{1 + \frac{4}{ab}} + \binom{n}{2} \left(\sqrt{1 + \frac{4}{ab}}\right)^2 + \binom{n}{3} \left(\sqrt{1 + \frac{4}{ab}}\right)^3 + \dots + 1 \\ & - \binom{n}{1} \sqrt{1 + \frac{4}{ab}} + \binom{n}{2} \left(\sqrt{1 + \frac{4}{ab}}\right)^2 - \binom{n}{3} \left(\sqrt{1 + \frac{4}{ab}}\right)^3 + \dots \end{aligned} \right]$$

$$= \frac{ab^{1-\chi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1}} \times \frac{(ab)^n}{2^{n-1}} \left[\binom{n}{0} + \binom{n}{2} \left(\sqrt{1 + \frac{4}{ab}}\right)^2 + \binom{n}{4} \left(\sqrt{1 + \frac{4}{ab}}\right)^4 + \dots \right].$$

Case: 1 If we consider $n = 2k$ then it follows that $\chi(n) = 0 \Rightarrow b^{1-\chi(n)} = b$.

$$\therefore G_{2k} = \frac{ab}{(ab)^{k+1}} \times \frac{(ab)^{2k}}{2^{2k-1}} \times \left[\binom{2k}{0} + \binom{2k}{2} \left(\sqrt{1 + \frac{4}{ab}}\right)^2 + \binom{2k}{4} \left(\sqrt{1 + \frac{4}{ab}}\right)^4 + \dots \right]$$

$$= \frac{(ab)^k}{2^{2k-1}} \sum_{r=1}^{\infty} \binom{2k}{2r} \left(\sqrt{1 + \frac{4}{ab}}\right)^{2r}$$

$$= \frac{(ab)^k}{2^{2k-1}} \sum_{r=1}^{\infty} \binom{2k}{2r} \left(1 + \frac{4}{ab}\right)^r.$$

Case: 2 If $n = 2k + 1$ then $\chi(n) = 1 \Rightarrow b^{1-\chi(n)} = 1$.

$$\therefore G_{2k+1} = \frac{a(ab)^{2k+1}}{(ab)^{k+1} 2^{2k}} \times \left[\binom{2k+1}{0} + \binom{2k+1}{2} \left(\sqrt{1 + \frac{4}{ab}}\right)^2 + \binom{2k+1}{4} \left(\sqrt{1 + \frac{4}{ab}}\right)^4 + \dots \right]$$

Proof: From the extended Binet's formula for G_n , we have

$$G_n = \frac{ab^{1-\chi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1}} (\alpha^n + \beta^n)$$

$$= c(\alpha^n + \beta^n); \text{ where } c = \frac{ab^{1-\chi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1}}$$

$$= \frac{c}{2^n} \left[\left(ab + \sqrt{a^2b^2 + 4ab}\right)^n + \left(ab - \sqrt{a^2b^2 + 4ab}\right)^n \right]$$

$$= \frac{c(ab)^n}{2^n} \left[\left(1 + \sqrt{1 + \frac{4}{ab}}\right)^n + \left(1 - \sqrt{1 + \frac{4}{ab}}\right)^n \right]$$

$$= \frac{a(ab)^k}{2^{2k}} \sum_{r=1}^{\infty} \binom{2k+1}{2r} \left(\sqrt{1 + \frac{4}{ab}}\right)^{2r}$$

$= \frac{a(ab)^k}{2^{2k-1}} \sum_{r=1}^{\infty} \binom{2k+1}{2r} \left(1 + \frac{4}{ab}\right)^r$, which completes the proof.

The following result expresses G_n explicitly in terms of α .

Theorem 2.3: $G_n = \left[c\alpha^n + \frac{1}{2} \right]$; where $c = \frac{ab^{1-\chi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1}}$ and $n \geq 2$.

Proof: We have $G_n = c(\alpha^n + \beta^n)$; where $c = \frac{ab^{1-\chi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1}}$. Then

$$|G_n - c\alpha^n| = |c|\beta^n| = \frac{c(ab)^n}{2^n} \left| 1 - \sqrt{1 + \frac{4}{ab}} \right|^n.$$

First we consider $ab \leq 4$. In this case for $n = 2$, we get the following values.

a	1	1	1	1	2	2	3	4
b	4	3	2	1	2	1	1	1
$c\alpha^n + \frac{1}{2}$	6.3284	5.2913	4.2321	3.1180	6.3284	4.2321	5.2913	6.3284

Thus the result holds true for such cases.

Now if $n > 2$, then we get $\left|1 - \sqrt{1 + \frac{4}{ab}}\right|^n < \frac{1}{2}$. So

$$|G_n - c\alpha^n| < \frac{c(ab)^n}{2^{n+1}} = \frac{ab^{1-\chi(n)}}{(ab)^{\lfloor \frac{n}{2} + 1 \rfloor}} \times \frac{(ab)^n}{2^{n+1}}.$$

Here if $n = 2k$, then $|G_n - c\alpha^n| < \frac{ab}{(ab)^{(k+1)}} \times \frac{(ab)^{2k}}{2^{2k+1}} \leq \frac{1}{2}$ as $ab \leq 4$.

Also if $n = 2k + 1$, then $|G_n - c\alpha^n| < \frac{a}{(ab)^{(k+1)}} \times \frac{(ab)^{2k+1}}{2^{2(k+1)}} \leq \frac{1}{2}$.

That is for $b \leq 4$, $|G_n - c\alpha^n| < \frac{1}{2}$.

Now if $ab > 4$ then $1 + \frac{4}{ab} < \left(1 + \frac{2}{ab}\right)^2$, which gives $\left|1 - \sqrt{1 + \frac{4}{ab}}\right|^n < \left(\frac{2}{ab}\right)^n$. Thus $|G_n - c\alpha^n| < c$.

Again, if $n = 2k$, then $|G_n - c\alpha^n| < \frac{1}{(ab)^k} < \frac{1}{4^k} < \frac{1}{2}$.

Also if $n = 2k + 1$, then we get $|G_n - c\alpha^n| < \frac{1}{(ab)^{k+1}b} < \frac{1}{2}$

as $ab > 4$. Thus for $ab > 4$, $|G_n - c\alpha^n| < \frac{1}{2}$ holds.

Hence in any case, $G_n = \left[c\alpha^n + \frac{1}{2}\right]$; where $c = \frac{ab^{1-\chi(n)}}{(ab)^{\lfloor \frac{n}{2} + 1 \rfloor}}$.

3. Summary

In this article, we have defined the new class of generalized Lucas sequence $\{G_n^{L(a,b)}\}$ and obtained the extended Binet formula for it. We have also obtained recursive formula for $G_n^{L(a,b)}$ to obtain the approximate values of its successor

and predecessor. We have also shown the relation of this sequence with classical Fibonacci sequence and classical Lucas sequence.

Authorship Contribution

This work is done by myself under the guidance of my Ph.D. mentor Dr. Devbhadra V. Shah, who is also the co-author of this paper.

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Conflict of Interest

There is no conflict of interest.

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