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Application of Sumudo Transform and Homotopy Perturbation Method to Solve Nonlinear Differential Equations

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Abstract: In this paper the Sumudo transform is coupled with Homotopy Perturbation Method to have the numerical solution of some nonlinear differential equations. The basic concept of applying such technique is to handled the nonlinear terms by using the He's polynomial.

Keywords: Sumudo Transform, Homotopy Perturbation Method, Logistic Differential Equation.

1. INTRODUCTION

In the literature survey we can observed the nonlinear phenomenon in many scientific and engineering application which are in the form of ordinary differential equations and partial differential equations. There are several techniques to solve such nonlinear ordinary differential equations and partial differential equations. Mahmoud S. Rawashdeh and Shehu Maitama [2] have introduced Natural Decomposition Method to solve nonlinear ordinary differential equations. There are other techniques to handle these nonlinear ordinary differential equations and partial differential equations. However such techniques needs large number of numerical computations. This paper deals with the method in which the Sumudo transform and Homotopy Perturbation Method are combined to solve some nonlinear differential equations.The nonlinear terms that occurs in the equations are decomposed using He's Polynomials.[1]

Recently Hassan Eltayeb and Adem Kilicman have developed the Sumudo Transform Method to solve the nonlinear system of partial differential equations and Volterra integro-differential equations.[4, 5] Jagdev Singh,

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Devendra Kumar and Adem Kilicman [6] have introduced the combined Homotopy Perturbation Method and Sumudu Transform to solve nonlinear Fractional Gas Dynamics equations.In this paper the numerical solution of logistic differential equation and Lokta-Volterra predator-prey model for the single species is obtained using the Sumudo Homotopy Perturbation Method(SHPM).

In early 90's Watugala [3] have introduced new integral transform the Sumudo transform which is defined over the set of functions

$A = [f(t)/\exists \text{ M}, \tau_1,\ \tau_2 > 0, \left|\text{f}(t)\right| < Me^{\frac{|t|}{\tau_j}}]$, if $t \in (-1)^j \times [0,\ \infty)]$ by the formula

$$\mathbb{S}[f(t)] = G(u) = \int_0^{\infty} e^{-t} f(ut)\,dt \quad u \in (-\tau_1,\ \tau_2) \tag{1}$$

Further details and properties about sumudo tranform can be seen in [7, 8, 9, 10].

1.1 Some Standard Result of Sumudo Transform

In this section we assume that all the considered functions are such that their Sumudo tranform exists.[9]

1. $\mathbb{S}[1] = 1$
2. $\mathbb{S}[\text{t}] = \text{u}$
3. $\mathbb{S}\left[\dfrac{t^{n-1}}{(n-1)!}\right] = u^{n-1}$ for $\text{n} = 1, 2, ...$
4. $\mathbb{S}[e^{at}] = \dfrac{1}{1-au}$
5. $\mathbb{S}\left[\dfrac{\sin(at)}{a}\right] = \dfrac{u}{1+s^2u^2}$
6. $\mathbb{S}[\cos(at)] = \dfrac{1}{1+s^2u^2}$
7. $\mathbb{S}\left[\dfrac{t^{n-1}e^{at}}{(n-1)!}\right] = \dfrac{u^{n-1}}{(1-au)^2}$
8. $\mathbb{S}[f(n)(t)] = \dfrac{1}{u^n}.G(u) - \sum_{k=0}^{n-1} \dfrac{f^k(0)}{u^{n-k}}$

2. SOLUTION OF NON-LINEAR EQUATIONS (PART-I)

The aim of this section is to discuss the use of sumudo transform algorithm to solve some nonlinear differential equations. To illustrate the basic technique, consider the following second order non-homogeneous nonlinear differential equation with the given initial condition of the form

$$Dx(t)+Rx(t)+Nx(t)=g(t) \tag{2}$$

where $x(0) = A$, $x'(0) = B$ are constants, $D=\frac{d^2}{dt^2}$ is the second order differential operator, R is the remaining linear operator, N is the general nonlinear differential operator and g(t) is a source term.
First apply the sumudo transform on both sides of the equation (2) we get,

$$\mathbb{S}[Dx(t)]+\mathbb{S}[Rx(t)]+\mathbb{S}[Nx(t)]=\mathbb{S}[g(t)] \tag{3}$$

Using the properties of sumudo transform we have,

$$\mathbb{S}[x(t)]=x(0)-x'(0).u+u^2\mathbb{S}[g(t)-Nx(t)-Rx(t)] \tag{4}$$

Now apply inverse Sumudo transform on both sides, we get

$$x(t)=x(0)-x'(0).t+\mathbb{S}^{-1}\left[u^2\mathbb{S}[g(t)-Nx(t)-Rx(t)]\right] \tag{5}$$

To find the exact solution x(t) we apply the Homotopy Perturbation Method for which consider $x(t)=\sum_{n=0}^{\infty}P^n x_n(t)$ and decompose the nonlinear term into the form $Nx(t)=\sum_{n=0}^{\infty}P^n H_n(t)$ where Hn(t) are the He's polynomial which can be calculated by the formula

$$H_n=\frac{1}{n!}\frac{d^n}{dP^n}\left(N\left(\sum_{n=0}^{\infty}P^n X_n\right)\right)n=0,\ 1,\ 2... \tag{6}$$

$\therefore$ Equation (5)becomes

$$\sum_{n=0}^{\infty}P^n x_n(t)=x(0)-x'(0).t+\mathbb{S}^{-1}\begin{bmatrix}u^2\mathbb{S}[g(t)]-p\mathbb{S}^{-1}[u^2\mathbb{S}] \\ \left[-\sum_{n=0}^{\infty}P^n H_n(t)-\sum_{n=0}^{\infty}P^n x_n(t)\right]\end{bmatrix} \tag{7}$$

From this last equation we have the recursive relation as

$$x_0(t) = H(t) = x(0) - x'(0)t + \mathbb{S}^{-1}\left[u^2.\mathbb{S}\left[g(t)\right]\right] \ \ and \tag{8}$$

$$x_{n+1}(t) = -\mathbb{S}^{-1}\left[R\sum_{n=0}^{\infty} P^n x_n(t) + \sum_{n=0}^{\infty} P^n H_n(t)\right] \tag{9}$$

This gives us the series solution of the given second order non-homogeneous nonlinear differential equation with the given initial condition.

2.1 Analysis of Method

Consider the logistic differential equation for the growth of population of single species of the form [12]

$$\frac{dP}{dt} = r.P\left[1 - \frac{P}{k}\right] \tag{10}$$

Where r and k are constants and P = P(t) represents the population of species at time t and $r\left[1 - \frac{P}{k}\right]$ is the per capita growth rate, k is the carrying capacity of the environment. Suppose that

$$X(r) = \frac{P(t)}{k}, \quad \omega = rt$$

which gives

$$\frac{dX}{d\omega} = X(1 - X)$$

with initial condition $X(0) = \frac{P_0}{k}$ where $P_0 = P(0)$.

Apply the sumudo transform on both sides we get

$$\mathbb{S}\left[\frac{dX}{d\omega}\right] = \mathbb{S}\left[X(1 - X)\right]$$

Using the properties of sumudo transform we have

$$\mathbb{S}\left[X(\omega)\right] = \frac{P_0}{x} + u.\mathbb{S}\left[X - X^2\right] \tag{11}$$

Apply the inverse sumudo transform on both sides

$$X(\omega)=\frac{P_0}{k}+\mathbb{S}^{-1}\left[u.\mathbb{S}\left[X-X^2\right]\right] \tag{12}$$

Now applying the classical Homotopy perturbation technique for which consider the solution of given logistic differential equation of the form

$$X(\omega)=\sum_{n=0}^{\infty}P^n X_n(\omega)$$

so that the equation (12) becomes

$$\sum_{n=0}^{\infty}P^n X_n(\omega)=\frac{P_0}{k}+P.\mathbb{S}^{-1}\left[u.\mathbb{S}\left[\sum_{n=0}^{\infty}P^n X_n(\omega)-\sum_{n=0}^{\infty}P^n H_n(\omega)\right]\right]$$

Where $H_n(\omega)$are the He's polynomial which can be calculated using the formula

$$H_n(X)=\frac{1}{n!}\frac{d^n}{dP^n}\left(N\left(\sum_{n=0}^{\infty}P^n X_n\right)\right) \quad n=0,\ 1,\ 2...$$

Now equating the terms with identical powers of P, we get

$$P^0 : X_0(\omega)=\frac{P_0}{k}$$

for the numerical purpose suppose that $P_0 = 3, k = 1$ so that $\frac{P_0}{k}=3$

$$P^1 : X_1(\omega)=\mathbb{S}^{-1}\left[u.\mathbb{S}[X_0-H_0]\right]=\left(\frac{P_0}{k}-\frac{P_0^2}{k^2}\right)\omega$$

$$P^1 : X_1(\omega)=-6\omega$$

$$P^2 : X_2(\omega)=\mathbb{S}^{-1}\left[u.\mathbb{S}[X_1-H_1]\right]=15\omega^2$$

$$P^3 : X_3(\omega)=\mathbb{S}^{-1}\left[u.\mathbb{S}[X_2-H_2]\right]=-37\omega^2$$

$$P^4 : X_4(\omega)=\mathbb{S}^{-1}\left[u.\mathbb{S}[X_3-H_3]\right]=\frac{365}{4}\omega^4$$

Continue in this way we get the series solution of the logistic differential equation in the form

$$\begin{aligned}X(\omega)&=X_0+X_1+X_2+X_3+\ X_4+X_5+X_6...\\&=3+(-6\omega)+15\omega^2+\left(-37\omega^3\right)+\frac{365}{4}\omega^4...\end{aligned}$$

3. SOLUTION OF NON-LINEAR EQUATIONS (PART-II)

Now consider the Lokta-Volterra system which is an integrating species predator-prey model governed by [11, 12, 13]

$$\frac{dN}{dt} = N[a - bP] \tag{13}$$

$$\frac{dP}{dt} = P[cN - d] \tag{14}$$

where a,b,c,d are constants N = N(t), P = P(t) are the prey predator population at time t respectively.
Suppose that

$$X(r) = \frac{c}{d}.N(t), Y(r) = \frac{b}{a}.P(t) \quad \omega = rt,\ \alpha = \frac{d}{a}$$

So that equation (13), (14) becomes

$$\frac{dX}{d\omega} = X(1 - Y)$$

$$\frac{dY}{d\omega} = \alpha.Y(X - 1)$$

with initial condition $X(0) = \delta$, $Y(0) = \beta$
Apply the sumudo transform on both sides we get

$$\mathbb{S}\left[\frac{dX}{d\omega}\right] = \mathbb{S}[X(1 - Y)]$$

Using the properties of sumudo transform we have

$$\mathbb{S}[X(\omega)] = \delta + u.\mathbb{S}[X(1 - Y)]$$

Apply the inverse sumudo transform on both sides

$$X(\omega) = \delta + \mathbb{S}^{-1}[u.\mathbb{S}[X(1 - Y)]$$

Similarly we have

$$Y(\omega) = \beta + \mathbb{S}^{-1}[u.\alpha.\mathbb{S}[Y(X - 1)]]$$

Let $\delta = 1.5, \beta = 0.8, \alpha = 1$

$$X(\omega) = 1.5 + \mathbb{S}^{-1}[u.\mathbb{S}[X(1 - Y)]]$$

$$X(\omega) = 0.8 + \mathbb{S}^{-1}\left[u.\mathbb{S}\left[Y\left(X-1\right)\right]\right]$$

Now applying the classical homotopy perturbation technique for that consider the solution of the form

$$X\left(\omega\right) = \sum_{n=0}^{\infty} P^n X_n\left(\omega\right)$$

$$Y\left(\omega\right) = \sum_{n=0}^{\infty} P^n Y_n\left(\omega\right)$$

$$\sum_{n=0}^{\infty} p^n X_n\left(\omega\right) = 1.5 + P.\mathbb{S}^{-1}\left[u.\mathbb{S}\left[\sum_{n=0}^{\infty} P^n X_n\left(\omega\right) - \sum_{n=0}^{\infty} p^n H_n\left(\omega\right)\right]\right]$$

$$\sum_{n=0}^{\infty} p^n Y_n\left(\omega\right) = 0.8 + P.\mathbb{S}^{-1}\left[u.\mathbb{S}\left[\sum_{n=0}^{\infty} P^n H_n\left(\omega\right) - \sum_{n=0}^{\infty} p^n Y_n\left(\omega\right)\right]\right]$$

where $H_n(\omega)$are the He's polynomial which can be calculated using the formula

$$H_n\left(X\right) = \frac{1}{n!}\frac{d^n}{dP^n}\left(N\left(\sum_{n=0}^{\infty} P^n X_n\right)\right) n = 0,\ 1,\ 2...$$

Now equating the terms with identical powers of P, we get

$$P^0 : X_0\left(\omega\right) = 1.5$$

$$\begin{aligned} P^1 : X_1\left(\omega\right) &= \mathbb{S}^{-1}\left[u.\mathbb{S}\left[X_0 - H_0\right]\right] \\ &= 0.3\omega \end{aligned}$$

$$\begin{aligned} P^2 : X_2\left(\omega\right) &= \mathbb{S}^{-1}\left[u.\mathbb{S}\left[X_1 - H_1\right]\right] \\ &= -0.27\omega^2 \end{aligned}$$

$$\begin{aligned} P^3 : X_3\left(\omega\right) &= \mathbb{S}^{-1}\left[u.\mathbb{S}\left[X_2 - H_2\right]\right] \\ &= -0.195\omega^3 \end{aligned}$$

$$P^0 : Y_0\left(\omega\right) = 0.8$$

$$\begin{aligned} P^1 : X_1\left(\omega\right) &= \mathbb{S}^{-1}\left[u.\mathbb{S}\left[H_0 - Y_0\right]\right] \\ &= 0.4\omega \end{aligned}$$

$$\begin{aligned} P^2 : Y_2\left(\omega\right) &= \mathbb{S}^{-1}\left[u.\mathbb{S}\left[H_1 - Y_1\right]\right] \\ &= 0.22\omega^2 \end{aligned}$$

$$P^3 : Y_3(\omega) = \mathbb{S}^{-1}\left[u.\mathbb{S}\left[H_2 - Y_2\right]\right]$$
$$= 0.03166\omega^3$$

continue in this way we get the series solution of the form

$$X(\omega) = X_0 + X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \ldots$$
$$= 1.5 + (0.3\omega) + \left(-0.27\omega^2\right) + \left(-0.195\omega^3\right)\ldots$$

$$Y(\omega) = Y_0 + Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 \ldots$$
$$= 0.5 + (0.4\omega) + \left(0.22\omega^2\right) + \left(0.03166\omega^3\right) + \ldots$$

4. CONCLUSION

This paper provides a new technique for the solution of nonlinear equations. Also we have obtained the series solution of logistic differential equation and Lokta-Volterra system which is an integrating species predator-prey model using this technique. The advantage of this technique is to handle the nonlinear term and reduce the computational work.

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