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Infinite Nested Radicals - A Way to Express All Quantities Rational, Irrational Transcendental By a Single Integer Two

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ABSTRACT

This paper proves that all mathematical quantities including fractions, roots or roots of root, transcendental quantities can be expressed by continued nested radicals using one and only one integer 2. A radical is denoted by a square root sign and nested radicals are progressive roots of radicals. Number of terms in the nested radicals can be finite or infinite. Real mathematical quantity or its reciprocal is first written as cosine of an angle which is expanded using cosine angle doubling identity into nested radicals finite or infinite depending upon the magnitude of quantity. The finite nested radicals has a fixed sequence of positive and negative terms whereas infinite nested radicals also has a sequenceof positive and negative terms but the sequence continues infinitely. How a single integer 2 can express all real quantities, depends upon its recursive relation which is unique for a quantity. Admittedly, there are innumerable mathematical quantities and in the same way, there are innumerable recursive relations distinguished by combination of positive and negative signs under the radicals. This representation of mathematical quantities is not same as representation by binary system where integer two has powers 0, 1, 2, 3…so on but in nested radicals, powers are roots of roots.

1. Introduction

A real mathematical quantity or its reciprocal can be expressed as cosine of an angle. Cosine of an angle can be written as double of its own angle by identity

$$\cos(x)=\left\{\frac{1+\cos(2x)}{2}\right\}^{\frac{1}{2}}=\frac{1}{2}\{2+2\cos(2x)\}^{\frac{1}{2}}$$

where x is an angle in radians. In above identity, cos($2x$) appears in right hand side and this cos($2x$) using the same identity, can be expressed in cos($4x$) and cos($4x$) in cos($8x$) so on and cos($2^{k-1}x$) in cos($2^{k}x$) where is k is any integer.

$$\cos(x)=\left\{\frac{1+\cos(2x)}{2}\right\}^{\frac{1}{2}}=\frac{1}{2}\{2+2\cdot\cos(2x)\}^{\frac{1}{2}},$$

$$\cos(2x)=\left\{\frac{1+\cos(4x)}{2}\right\}^{\frac{1}{2}}=\frac{1}{2}\{2+2\cdot\cos(4x)\}^{\frac{1}{2}},$$

$$\cos(4x)=\left\{\frac{1+\cos(8x)}{2}\right\}^{\frac{1}{2}}=\frac{1}{2}\{2+2\cdot\cos(8x)\}^{\frac{1}{2}},$$

$$\cos\left(2^{k-1}x\right)=\left\{\frac{1+\cos\left(2^{k}x\right)}{2}\right\}^{\frac{1}{2}}=\frac{1}{2}\left\{2+2\cos\left(2^{k}x\right)\right\}^{\frac{1}{2}}.$$

or

$$\cos\left(2^{k-1}x\right)=\sqrt{\frac{1+\cos\left(2^{k}x\right)}{2}}=\frac{1}{2}\sqrt{2+2\cos\left(2^{k}x\right)}$$

From these identities, it can be deduced, where k is any integer.

$$\cos(x)=\frac{1}{2}\cdot\sqrt{2+(or\,-)\sqrt{2+(or\,-)\ldots\ldots\sqrt{2+(or\,-)2\cdot\cos\left(2^{k}\cdot x\right)}}}$$

Angle x on being doubled continuously, there comes a stage when cos($2^{k}x$) equals cos(x) or –cos(x). How and why that stage comes, will be analyzed and formula given as the paper proceeds. At that stage, right hand side will contain cos(x) which is same as in left hand side. Now cos(x) in right hand side can be replaced by all nested radicals prior to and including cos($2^{k}x$). In other words, all terms from cos($2x$) to cos($2^{k}x$) can be substituted for cos($2^{k}x$) which equals cos(x) or – cos(x). Therefore, on successively putting value of cos(x) in right hand side, equation proceeds infinitely and takes the form (Landau, 1992).

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$$\cos(x)=\frac{1}{2}\cdot\sqrt{2+(or-)\sqrt{2+(or-)\ldots\ldots\sqrt{2+(or-)2\cdot\cos(x)}}}$$

In the above equation, both positive and negative signs are written to indicate that one sign out of the two depending upon the sign of magnitude of cos(2*x*), cos(4*x*), cos(8*x*) or $\cos(2^k x)$ will be applicable. Above equation is recursive in nature as cos(*x*) appears both in left and right hand side. On successively substituting the value of cos(*x*), equation takes the form.

$$\cos(x)=\frac{1}{2}\cdot\sqrt{2+(or-)\sqrt{2+(or-)\sqrt{2+(or-)\sqrt{2+or-\ldots}}}}$$

Sign …written in above nested radicals (Weisstein, Eric) denotes that this nested radical extends infinitely. Angle *x* is known from magnitude of quantity being expressed in continuous nested radicals, signs positive or negative of cos(2*x*), cos(4*x*), cos(8*x*), ... etc can also be known from value of angle *x* and will be mentioned accordingly in the above equation.

2. Theory and Concept

With this background, it is known that cos(*x*) can be written in infinite nested radicals using number 2. Naively, it appears that for whatsoever value of *x*, cos(*x*) can be written in infinite nested radicals but what matters is the sign (positive or negative) of cos(2*x*), cos(4*x*), cos(8*x*) etc. or one can say positive or negative signs of $\cos(2^k x)$ where k is 1, 2, 3, … However, depending upon the value of *x*, there may be cases where cos(*x*) can be written in finite nested radicals, it will be discussed as the paper proceeds.

Cos(*x*) is positive in first (0 to 90 degrees) quadrant and fourth (270 to 360 degrees) quadrant. Accordingly, signs of $\cos(2^k x)$ can be determined in accordance with magnitude of angle. If angle falls in 2nd and 3rd quadrants, cosine of that angle will be negative otherwise positive and accordingly signs of nested radicals can be written. For example, if $x = \pi/3$, then $\cos(2\pi/3) = -\cos(\pi/3)$ that is negative. Here RHS contains $-\cos(\pi/3)$ and LHS contains $\cos(\pi/3)$ and recurrence takes place. It is exemplified below.

$$\cos(\pi/3)=\frac{1}{2}\left\{2+2\cos\left(\frac{2\pi}{3}\right)\right\}^{\frac{1}{2}}=\frac{1}{2}\left\{2-2\cos\left(\frac{\pi}{3}\right)\right\}^{\frac{1}{2}} \quad (1)$$

Now $-\cos(\pi/3)$ in RHS of equation (1) can be written as equal to $-\frac{1}{2}\left\{2-2\cos\left(\frac{\pi}{3}\right)\right\}^{\frac{1}{2}}$. On substituting this value, equation (1) transforms to

$$\cos(\pi/3)=\frac{1}{2}\times\left[2-\left\{2-2\cos\left(\frac{\pi}{3}\right)\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

On successive substitution, the equation takes the form.

$$\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\ldots}}}}}$$

It is clear from above that recurrence in this case takes place at $\cos(2^k\cdot\pi/3)$ where k equals 1 and recursive relation of signs is minus (–) as shown below.

$$\cos(\pi/3)=\frac{1}{2}\left\{2-2\cos\left(\frac{\pi}{3}\right)\right\}^{\frac{1}{2}}$$

In the above case, k was 1 and recurrence found easily but there may be cases where k being large, may be a bit difficult to find.

Next task is how to find k for recursive relation to take place. For this purpose, mathematical quantities will be subdivided into three categories. Before categorization, it is submitted, all mathematical quantities can be expressed as $\cos(\pi/n)$ or $\cos\left[\frac{\pi}{(p/q)}\right]$ where n is any integer or even a fraction of the form *p*/*q* where p and q are integers provided these mathematical quantities lie in the domain of –1 to +1 both –1 and +1 inclusive. However, the mathematical quantities are not limited to the range of –1 to +1 and may extend from minus 1 to minus infinity or plus 1 to plus infinity and per se can not be represented by cos(*x*) (as it is) which has range of –1 to +1. But these quantities can always be brought down to the range of –1 to +1 if reciprocal of these quantities are considered. If a mathematical quantity is >1 or it is <–1 then for normalizing these to the range of –1 to +1 we can write

$$\cos(x)=\frac{1}{\text{mathematical quantity}}$$

and once in the range of –1 to +1, nested radicals can be found. Thereafter, reciprocal of nested radicals of cos(*x*) will equal original mathematical quantity. Coming to categorization of quantities, categories can be classified into three sets of mathematical quantities.

1) Quantities of the type which can be expressed as $\cos(\pi/n)$ or $\cos\left[\frac{\pi}{(p/q)}\right]$ where n (or p) is not divisible by 2 i. e. n is an odd integer and if n is a fraction of the form p/q, then p is an odd integer.
2) Quantities of the type $\cos(\pi/n)$ or $\cos\left[\frac{\pi}{(p/q)}\right]$ where n (or p) is divisible by 2 and is of the form $s\cdot 2^r$ where s is an odd integer (not 1) and r is any integer 1, 2, 3,….
3) Quantities of the type $\cos(\pi/n)$ or $\cos\left[\frac{\pi}{(p/q)}\right]$ where n (or p) is divisible by 2 i.e n (or p) is even integer but is of the form 2^r where r is an integer 1, 2, 3,…

2.1 Category 1 Mathematical quantities $\cos(\pi/n)$ or $\cos\left[\frac{\pi}{(p/q)}\right]$ of category 1 where n is odd integer or n is a fraction of form p/q where p is odd integer.

2.1.a. Proposition

If n is any integer but not multiple of 2 or n is of the form p/q where p is not multiple of 2 then $(2^k + 1)$ or $(2^k - 1)$ will always be divisible by n (or p) for some integer value of k.

It is obvious $(2^k + 1)$ and $(2^k - 1)$ will always be odd on account of the fact that 2^k is even integer and if 1 is added or subtracted, resultant quantity will always be odd. Also division of an odd integer by some other odd integer without remainder, can be possible.

Here k can assume any value from 1, 2, 3,… To find out the value of k, it is first assigned value 1 and checked whether $(2^k + 1)$ or $(2^k - 1)$ is fully divisible by n. If it is not, then k is taken as 2, then 3….so on till that value of k is reached when n (or p) divides without any remainder. There is thus a positive integers available for k to satisfy above equation. Since odd can be divided by odd as stated earlier, there is a complete possibility of division of $(2^k + 1)$ or $(2^k - 1)$ by n without any remainder. Let

$$m = (2^k + 1)/n \qquad (2)$$

or

$$m = (2^k - 1)/n \qquad (3)$$

as the case may be, where m is that minimum number where $(2^k + 1)$ or $(2^k - 1)$ is just divisible by n without remainder.

2.1.b. Application of Proposition

Applying this proposition that there exists a value of k that makes $(2^k + 1)$ or $(2^k - 1)$ fully divisible by n, then k_{th} term of nested radicals for $\cos(x)$ will be $\cos(2^k \times \pi/n)$ where recurrence takes place and

$(2^k+1)/n = m$ or $(2^k-1)/n = m$

First taking the case where $(2^k+1)/n = m$ and multiplying both sides by π and rearranging, makes $2^k\cdot(\pi/n) = (m\pi - \pi/n)$ or

$\cos\{2^k\cdot(\pi/n)\} = \cos(m\pi - \pi/n) = -\cos(\pi/n)$ when m is odd and

$\cos\{2^k\cdot(\pi/n)\} = \cos(m\pi - \pi/n) = \cos(\pi/n)$ when m is even.

For the cases where $(2^k-1)/n = m$ and adopting same procedure,

$\cos\{2^k\cdot(\pi/n)\} = \cos(m\pi + \pi/n) = -\cos(\pi/n)$ when m is odd and

$\cos\{2^k\cdot(\pi/n)\} = \cos(m\pi + \pi/n) = \cos(\pi/n)$ when m is even.

That means recursive relation will happen when relation (2) or (3) is satisfied and the term $\cos\{2^k\cdot(\pi/n)\} = -\cos(\pi/n)$, or $\cos(\pi/n)$. When this stage is reached, signs positive or negative of subsequent terms will follow the recursive relation. The situation will get further clarified while dealing with following examples.

2.1.c. Examples

Example 1

Infinite nested radicals for $\cos(\pi/7)$. Here n is 7 and it satisfies $(2^3-1)/7 = 1$, its term $\cos\{2^k\cdot(\pi/n)\}$ will correspond to $\cos\{2^3\cdot(\pi/7)$ i.e. it will have three terms for recursive relation to take place. In this case,

$\cos(2\pi/7)$ will be positive,

$\cos(4\pi/7) = -\cos(3\pi/7)$ (negative)

$\cos(8\pi/7 = -\cos(\pi/7)$ (negative)

Therefore, recursive relation of signs will be $+--$ and these signs will repeat infinitely as shown below.

$$+--+--+--+--+--+--+--+--\ldots up\ to\ \infty.$$

$$\cos\left(\frac{\pi}{7}\right) = \frac{1}{2}\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\ldots}}}}}}$$

Example 2.

Infinite nested radicals for $\cos(\pi/15)$ and $\cos(\pi/17)$

For $\cos(\pi/15)$ n is 15. It satisfies $(2^4 - 1)/15 = 1$, its terms $\cos\{2^k\cdot(\pi/n)\}$ will correspond to $\cos\{2^4\cdot(\pi/15)\}$ i.e. it will have four terms for recursive relation to take place.

$\cos(2\pi/15) = +ve$, $\cos(4\pi/15) = +ve$,

$$\cos(8\pi/15) = -ve,$$

$$\cos(16\pi/15) = -\cos(\pi/15) = -ve$$

Since, $-\cos(\pi/15)$ appears in RHS and equals in magnitude to LHS, recurrence takes place at this stage.

Recursive relation of signs is $++--$. Sign of first term which is positive is omitted and nested radicals will proceed with signs $++--++--++--\ldots$ so on up to infinity and nested radicals will be as given below.

$$\cos\left(\frac{\pi}{15}\right) = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\ldots}}}}}$$

For $cos(\pi/17)$. Here n is 17. It satisfies $(2^4 + 1)/17 = 1$. Its term $cos\{2^k \times (\pi/n)\}$ will correspond to $cos\{2^4 \times (\pi/17)\}$ i.e. it will have four terms for recursive relation to take place.

$cos(2\pi/17) = +\text{ve}$, $cos(4\pi/17) = +\text{ve}$

$$cos(8\pi/17) = +\text{ve},$$

$$cos(16\pi/17) = -cos(\pi/17) = -\text{ve}.$$

But since it equals LHS, recurrence takes place at this angle. Recursive relation of its signs is $+++-+++-\ldots$ so on up to infinity.

$$\cos\left(\frac{\pi}{17}\right) = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\ldots}}}}}$$

Nested radicals of cos π/15 differs from that of cos π/17 on account of the signs of recursive relations.

Recursive relation of signs of $cos\left(\frac{\pi}{15}\right)$ is $++--++--\ldots$ so on up to infinity and recursive relation of signs of $cos\left(\frac{\pi}{17}\right)$ is $+++-+++-\ldots$ so on up to infinity.

Example 3

Infinite nested radicals for $cos(4\pi/19)$.

$cos(4\pi/19)$ can be written as $cos\left(\frac{\pi}{19/4}\right)$ and that makes n as 19/4 which is of *p*/*q* form and it satisfies $(2^9+1)/(19/4) = 108$, its terms $cos\{2^9 \times (\pi/n)\}$ will correspond to $cos 2^9 \cdot \left(\frac{\pi}{19/4}\right)$ i.e. it will have nine terms for recursive relation to take place. Signs of terms of nested radicals will be

$$cos(8\pi/19) = +ve$$

$$cos(16\pi/19) = -cos(3\pi/19) = -ve,$$

$$cos(32\pi/19) = cos(6\pi/19) = +ve,$$

$$cos(64\pi/19) = -cos(7\pi/19) = -ve,$$

$$cos(128\pi/19) = -cos(5\pi/19) = -ve,$$

$$cos(256\pi/19) = -cos(9\pi/19) = -ve,$$

$$cos(512\pi/19) = -cos(\pi/19) = -ve,$$

$$cos(1024\pi/19) = cos(2\pi/19) = +ve,$$

$$cos(2048\pi/19) = cos(4\pi/19) = +ve.$$

At this stage, RHS contains cos(4π/19) left hand side, therefore, recursive relation of signs, (omitting sign of first term which is always positive) is $+-+----++$ Using recurrence, it will proceed infinitely as $+-+----+++-+----++\ldots$ so on up to infinity. Infinite nested radicals for cos 4π/19 is given below

$$\frac{1}{2}\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\ldots}}}}}}}}}$$

Example 4

Infinite nested radicals for $cos(\pi/19)$.

Using the data as found above and knowing that

$$cos(2\pi/19) = +\text{ve}$$

$cos(4\pi/19) = +ve$,

recursive relation is $+++-+----+++-+----+++-+----\ldots$ and it will proceed infinitely. $cos(\pi/19)$, therefore can be written in infinite nested radical as

$$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+.}}}}}}}}}}$$

Example 5

Infinite nested radicals for $cos\left(\frac{\pi}{81}\right)$. Following procedure as given in earlier examples, recursive relation is found as $+++++-+-+----+--++----+-++++-$. Recurrence of sign will occur as $+++++-+-+----+--++---+-+++-+++++-+-+----+--++---+-+++-\ldots\infty$. Infinite radicals for $cos\left(\frac{\pi}{81}\right)$ is given below.

$$\cos\left(\frac{\pi}{81}\right) = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+}}}}}}}}}}}}}}}}}}}}}}}}}}}}$$

2.2 Category 2) Quantities of the type cos(π/n) or $\cos\left(\frac{\pi}{(p/q)}\right)$ where n or p is divisible by 2 i.e. these are even integers and are of form $n = s \cdot 2^r$ where s is odd integer not one but n is not of the form 2^r

In such cases, n is even integer but is of form $s{\cdot}2^r$ where s is odd but not one and r is 1, 2, 3, 4,..., the proposition $(2^k+1)/n = m$ or $(2^k-1)/n = m$ is not applicable as 2^k+1 or 2^k-1 is always odd and will never be divisible by n which is even. But n though even has one or more factors which are odd and that makes it as $n = s \cdot 2^r$ where s is an odd integer but not one and r may be 1, 2, 3 or any other number depending upon the nature and magnitude of quantity n. In such quantities, nested radicals have 'r' non recurring or fixed terms and after first 'r' fixed terms, it has recurring terms corresponding to odd number s. That is, nested radicals will start with fixed terms and then has recurring terms. The proposition takes the form

$$(2^k+1)/s = m \text{ or } (2^k-1)/s = m$$

and is applicable as 2^k+1 or 2^k-1 are always odd and s being odd, can divide completely in following way

$$(2^k+1)/s = m \text{ or } (2^k-1)/s = m$$

That further leads to the result that recursive relation is applicable to the odd factor s. That means there exists a recursive relation but for odd factor s. In other words, nested radicals pertaining to 2^r are non recurring and belong to fixed part whereas other part relating to s recurs infinitely. The situation will further get clarified by the examples given below.

2.2.a. Examples

Example 1

Infinite nested radicals for $cos(\pi/36)$. Here n is 36 which can be written as $9{\cdot}(2^2)$. That is $r = 2$ and $s = 9$. That means fixed part has two terms and recurring part pertains to 9.

Fixed part

$$cos(2\pi/36) = +\text{ve}, \; cos(4\pi/36) = +\text{ve}$$

Recurring part

$$cos(8\pi/36) = +\text{ve}, \; cos(16\pi/36) = +\text{ve},$$

$$cos(32\pi/36) = +\text{ve}.$$

Therefore, nested radicals will have signs $++ \quad ++-$ where fixed part has signs $++$ and recurring part has signs $++-$ which will proceed infinitely as $++ \quad ++-++-++-\ldots up\,to\,\infty$.

Therefore, $cos(\pi/36)$ equals to

$$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-}}}}}}}\ldots\ldots\ldots\ldots$$

|< ---F.P.-- >|< ---R.P------ >|

where F.P. and R.P. denote fixed and recurring parts.

Example 2

Infinite nested radicals for $cos(\pi/100)$

Here n is 216 which can be written as $27(2^3)$. That is r = 3 and s = 27. That means fixed part has three terms and recurring part pertains to 27.

Fixed part

$cos(2\pi/216) = +\text{ve}$, $cos(4\pi/216) = +\text{ve}$,

$$cos(8\pi/216) = cos(\pi/27) = +\text{ve}.$$

Recurring part

$$cos(2\pi/27) = +\text{ve}, cos(4\pi/27) = +\text{ve}$$

$$cos(8\pi/27) = +\text{ve}, cos(16\pi/27) = -\text{ve}$$

$$cos(32\pi/27) = -\text{ve}, cos(64\pi/27) = +\text{ve}$$

$$cos(128\pi/27) = -\text{ve}, cos(256\pi/27) = -\text{ve}$$

$$cos(512\pi/27) = -cos(2\pi/27) = -\text{ve}$$

Therefore, nested radicals will have signs $+++ \quad +++--+--+---$ where fixed part has signs $+++$ and recurring part has signs $+++--+--+---$. Infinite nested radicals will proceed $++++++--+--+---++ +--+--+---\ldots up\,to\,\infty$.

Therefore, $cos(\pi/216)$ can be written as given below.

$$\cos(\pi/216) = \frac{1}{2}.\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\ldots\ldots\ldots}}}}}}}}}}}}}$$

< -FP------------ >|< --------------------RP------------------------------ >

Example 3

Infinite nested radicals for $cos(\pi/100)$.

Here n is 100 which can be written as $25(2^2)$. That is r = 2 and s = 25.

That means fixed part of infinite nested radicals has two terms and recurring part pertains to 25.

Fixed part

$$cos(2\pi/100) = +\text{ve},$$

$$cos(4\pi/100) = cos(\pi/25) = +\text{ve}.$$

Recurring part

$$cos(2\pi/25) = +\text{ve},$$
$$cos(4\pi/25) = +\text{ve}.$$
$$cos(8\pi/25) = +\text{ve},$$
$$cos(16\pi/25) = +\text{ve},$$
$$\cos(32\pi/25) = -\text{ve},$$
$$cos(64\pi/25) = -\text{ve},$$
$$cos(128\pi/25) = -\text{ve},$$
$$cos(256\pi/25) = +\text{ve},$$
$$cos(512\pi/25) = +\text{ve},$$
$$cos(1024\pi/25) = -cos(\pi/25) = -\text{ve}.$$

Therefore, nested radicals will have signs $++\ +++----++-$ where signs of fixed part are $++$ and signs of recurring part are $+++----++-$. Infinite nested radicals will proceed $+++++----+-+++-----++-\ldots$ *up to* ∞.
Therefore, $cos(\pi/216)$ equals to nested radicals given below.

$$\cos\pi/100 = \frac{1}{2}.\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\ldots\ldots}}}}}}}}}}}}}$$

< ---F.P------> |< --------------------------------R.P-------------------------------- >

2.3.Category 3 Quantities of the type $\cos(\pi/n)$ *or* $\cos\left[\frac{\pi}{(p/q)}\right]$ *where n or p is divisible by 2 i.e. these are even integers but are of the form* 2^r

In such cases, n is even integer and is of form 2*r* and it has all even factors, proposition $(2^k+1)/n = m$ or $(2^k-1)/n = m$ is not applicable as $(2^k+1)/n$ or $(2^k-1)/n$ is always odd and will never be divisible by n which is even. Since n is of form 2*r*, nested radicals will have only fixed part (Zimmerman & Ho, 2008) and there will be no recurring part. In these cases, there will not be infinite nested radicals but radicals are finite to the extent of r terms. The situation will further get clarified by the examples given below.

2.3.a. Examples

Example 1

Nested radicals for $cos(\pi/8)$ and François Viete formula for π (Rick, 2008).
Here n = 8 = 2^3.
$cos\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}} = +ve$ and cos (π/2) = 0. Therefore, nested radicals will stop at cos (π/2) = 0.

Therefore, (Zimmerman & Ho, 2008)
$cos(\pi/8) = \frac{1}{2}.\sqrt{2+\sqrt{2}}$.

Similarly, $cos\left(\frac{\pi}{16}\right) = \frac{1}{2}\times\sqrt{2+\sqrt{2+\sqrt{2}}}$

$$cos\left(\frac{\pi}{32}\right) = \frac{1}{2}\times\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$cos\left(\frac{\pi}{64}\right) = \frac{1}{2}\times\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} \ldots\ldots\ldots \textit{so on.}$$

Mathematician François Viete (Herschfeld, 1935) utilized these values for calculating the value of π from identity

$$\lim_{n\to\infty}\frac{\sin(x)}{x} = \cos\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2^2}\right)\cos\left(\frac{x}{2^3}\right)\ldots\cos\left(\frac{x}{2^n}\right)$$

$$\lim_{n\to\infty}\frac{\sin(x)}{x} = \cos\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2^2}\right)\cos\left(\frac{x}{2^3}\right)\ldots 1$$

Let $x = \frac{\pi}{4}$ then

$$\frac{4}{\pi\sqrt{2}}$$
$$= \left(\frac{1}{2}\cdot\sqrt{2+\sqrt{2}}\right)\cdot\left(\frac{1}{2}\cdot\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\left(\frac{1}{2}\cdot\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)\ldots$$

$$\frac{1}{\pi} = \frac{1}{2}\cdot\left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{1}{2}\cdot\sqrt{2+\sqrt{2}}\right)\left(\frac{1}{2}\cdot\sqrt{2+\sqrt{2+\sqrt{2}}}\right)$$
$$\cdot\left(\frac{1}{2}\cdot\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)\ldots$$

Sign … means terms are continuing up to infinity. Above is an equation that represents π, a transcendental quantity by nested radicals using only integer 2.

2.3.b. Expressing sin x in nested radicals

In the beginning, we considered the identity

$$\cos(x) = \frac{1}{2}\{2+2\cos(2x)\}^{\frac{1}{2}}$$

But here, identity sin(*x*), in terms of cos(2*x*) will be used as given below.

$$\text{Sin}(x) = \frac{1}{2}\{2-2\cos(2x)\}^{\frac{1}{2}}$$

Expansion of cos(2*x*) in infinite/finite nested radicals has already been explained. Therefore substituting infinite/finite radicals of cos(2*x*) in above identity, Sin(*x*) can be expressed in continuous infinite or finite radicals as the case may be.

2.4 Expression of any Quantity In Infinite/Finite Nested Radicals

1.Quantities Pertaining To First Category

Let there be any quantity N or it is of the form *p/q* where p and q are integers. Two cases arise, either modulus $|N|$ or $|p/q|$ is > 1 or it is < 1. Modulus of a quantity is its magnitude ignoring its sign, modulus of $-M$ will be M, modulus of M will be M. Modulus is denoted by two verticals lines with magnitude in between, modulus M is written as $|M|$. When $|N|$ or $|p/q|$ is >1 it can be brought down to a quantity less than 1 by taking its reciprocal and if it is less than 1, there does not arise necessity of taking its reciprocal. After bringing it down to value less than 1, $1/|N|$ or $1/|p/q|$ as the case may be is equated to $cos(\pi/\text{n})$.

That is

$1/N = cos(\pi/\text{n})$ or $q/p = cos(\pi/\text{n})$ when $|N| > 1$ or $|p/q| > 1$. Value of (π/n) is found out by taking cos inverse 1/N or cos inverse q/p as the case may be.

When $|N| < 1$ or $|p/q| < 1$, $N = \cos(\pi/n)$ or $p/q = \cos(\pi/n)$. From these equations, $N = \cos(\pi/n)$ or $p/q = \cos(\pi/n)$, value of (π/n) can be found and depending upon its value whether it falls in category 1, category 2 or category 3, it can be expanded in infinite or finite nested radicals accordingly.

Examples

Let N = 2, that means $|N| > 1$, therefore, we will take $\cos(\pi/n) = 1/N = 1/2$ or $(\pi/n) = (\pi/3)$. That makes n = 3 and $\cos(\pi/3)$ can be expanded in infinite nested radicals and will equal $1/N$.

Let $N = -2$, that means $|N| > 1$, therefore, we will take $cos(\pi/\text{n}) = 1/\text{N} = -1/2$ or $(\pi/n) = (2\pi/3)$. That makes $n = p/q = 3/2$ and $cos(2\pi/3)$ can be expanded in infinite nested radicals and will be equal to $1/N$.

Let $N = \left(5^{\frac{1}{2}} - 1\right)/4$. That means $|N| < 1$, therefore, we will take $cos(\pi/\text{n}) = \left(5^{\frac{1}{2}} - 1\right)/4$ or $(\pi/\text{n}) = (\pi/5)$.

That makes $n = 5$ and $cos(\pi/5)$ can be expanded in infinite nested radicals.

3. Quantities Pertaining to Second and Third Category

Let there be a quantity N or quantity of the form *p/q* again two cases arise, either $|N|$ or $|p/q|$ is >1 or it is < 1. When it is more than 1, it can be brought down to a quantity less than 1 by taking its reciprocal and if it is less than 1, there does not arise any necessity of taking its reciprocal. After bringing it down to value less than 1, here also, it will be equated with $\cos(\pi/n)$. Values of (π/n) can be found out and from that n can be calculated. In these cases n will be of the form $s{\cdot}2^r$ or 2^r. Nested radicals can be determined for these quantities as already explained,

3.1 Insight

All quantities can be divided into two categories, one which has modulus of their magnitude less than one and others which have greater than one. Those which have greater than one, will have their reciprocal less than one. Thus all quantities either directly or indirectly (by taking their reciprocal) can be made less than 1 and hence can be equated with cosine (or sine) of an angle. That angle can be determined by talking inverse and this angle can be expressed as (π/n) where value of n depends upon the magnitude of the quantity N. Once a quantity is expressible as $\cos(\pi/n)$, angle (π/n) canbe doubled successively by identity

$$cos(\pi/\text{n}) = \frac{1}{2}\cdot\left\{2 + 2\cos\left(2\frac{\pi}{n}\right)\right\}^{\frac{1}{2}}$$

In this process, a stage will come when RHS contains the term $cos(2^k \cdot \pi/\text{n})$ which equals to either $cos(\pi/\text{n})$ or $-cos(\pi/\text{n})$ where k and n satisfies the equation $(2^k + 1)/n = m$ or $(2^k - 1)/n = m$ then recursive relation is established and k nested radicals continue infinitely. It is reiterated k, n and m are all integers. It may also happen that n is even of the form $s{\cdot}2^r$ where *s* and *r* are integers but s is odd not one. In these cases, nested radicals will have fixed part corresponding to 2^r and recurring part corresponding to s satisfying one of the relations $(2^k + 1)/s = m$ or $(2^k - 1)/s = m$ for recurrence to take place.

If *n* found is of the form 2^r, then RHS will have fixed nested radicals as $cos(2^{\text{r}-1} \cdot \pi/\text{n})$ will be zero. Since identity $\cos(x) = \frac{1}{2}\{2 + 2\cos(2x)\}^{\frac{1}{2}}$ is used, therefore, it has all the terms containing integer 2. In this way, integer 2 can express all quantities and it is the combination of signs positive and negative of recurrence relation that decides magnitude of the quantity being expressed in nested radicals. Since a combination may consist of a number of positive and negative terms depending upon the magnitude and sign of the quantity, therefore, there may be infinite combinations of positive and negative terms of recursive relation and such infinite number of combinations will express quantities infinite in number.

4. Conclusions and Results

All quantities which are less than one, are expressible by cosine or sine of an angle and hence can be written in infinite or finite nested radicals depending upon the magnitude of the quantity. Quantities which have modulus more than one, their reciprocal are expressible by cosine or sine of an angle. By angle doubling identity, angle can be successively doubled till cosine of the resultant angle equals to positive or negative cosine of original angle. At that stage recurrence takes place and cosine of the original angle is substituted by the already found nested radicals. Since after substitution, cosine of original angle again appears in RHS, substitution is repeated infinitely. Since cosine angle doubling identity, involves integer 2 and only 2, but with different combinations of positive and negative signs depending upon the magnitude of the quantity, therefore, recursive relation of signs decides the magnitude. When a mathematical quantity is expressed as $cos\left(\frac{\pi}{s \cdot 2^r}\right)$ where s is odd integer but not one and r is any number 1, 2, 3, ….. so on, then infinite radicals has fixed part corresponding to r and recurring part corresponding to s. Fixed part appears once in the beginning whereas recurring part repeats infinitely. If a mathematical quantity is expressed as $cos\left(\frac{\pi}{2^r}\right)$ then nested radicals are finite in numbers (Zimmerman & Ho, 2008) and these do not repeat infinitely. Last, in this way 2 and only 2 is the integer that can represent all mathematical quantities by its various combination of signs of positive and negative of terms of infinite/finite nested radicals.

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