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On The Numerical Solutions of Boundary Layer Equations of Williamson Fluid Past a Moving Plate

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ABSTRACT

Laminar boundary layer flow of Williamson fluid over a moving plate is discussed in this paper. The governing equations of the flow problem are transformed into similarity equations using similarity technique. The reduced equations are numerically solved by finite difference method. The graphical presentation is discussed. Obtained results are compared with that available in literature.

1. Introduction

Mostly non-Newtonian fluids are types of pseudo-plastic fluids. Wide range of industrial application like suspension coated sheets (photographic films), high molecular weight polymers melts and its solutions etc… The boundary layer flows of such fluids are of great interest in commercial point of view. All the rheological properties of such types of fluids are not explained by the Navier-Stokes equations. Therefore, to overcome this type of deficiency, different types of fluid models has been proposed by researchers. The power-law model, Cross model, Carreaus model and Ellis model are its examples but very rare information available about the Williamson fluid model in literature.

In Williamson fluid viscosity decreases with increasing rate of shear stress due to this characteristics is known as a non-Newtonian fluid. Chyme in small intestine treated as Williamson fluid. The boundary layer in liquid film condensation process, in the industries extrusion of a polymer sheet from the die, emulsion coating on photographic films are examples of Williamsons fluid. Gastrointestinal tract in a human body, the peristaltic phenomenon plays a vital role throughout the digestion and absorption of food.

Williamson (1929) proposed a model for pseudo plastic materials when it moves and gives it equation of motion and also experimentally verified the results. Lyubimov and Perminov (2002) discussed in the presence of a gravitational field the flow of a thin layer over an inclined surface of a Williamson fluid. Dapra and Scarpi (2007) considered Williamson fluid injected into a rock fracture and gives the perturbation solution. Nadeem *et al.* (2010) discussed peristaltic flow of a Williamson fluid. Vasudev *et al.* (2010) discussed in details the peristaltic pumping of a Williamson fluid through a porous medium by considering heat transfer. Srivastava et al. (1995, 2007) discussed the flow in non-uniform geometry and gives theoretical model under the effects of an inserted endoscope on Chyme movement in small intestine. Nadeem et al. (2013) find the solution flow of a Williamson Fluid over a stretching sheet by applying homotopy analysis method (HAM). L.M. Srivastava et al. (1983, 1984) introduced Casson model for peristaltic transport of blood and a physiological fluid. I. Dapra et al.

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(2007) discussed perturbation solution for pustule flow of a non-Newtonian Williamson fluid in a rock fracture.

In the present paper, boundary layer flow of Williamson fluid over a moving plate is analyzed. The flow considered in two dimensional semi-infinite moving surface. Along the x-axis two opposite and equal forces are applied to produce stretching and keeping the fixed origin. The flow is generated due to the linear stretching. The governing equations of the flow problem are highly non-linear PDE. Therefore we have transformed the governing equations into non-linear ODE using similarity technique. The reduced equations are numerically solved by Finite difference method with MATLAB ODE Solver. The graphical presentation is also given with comparison.

2. Governing Equations

The continuity and momentum equations for an incompressible fluid flow of the Williamson Fluid are given by (Nadeem et al, 2013)

Continuity equation:

$$\nabla \cdot V = 0 \tag{1}$$

Momentum equation:

$$\rho \frac{dV}{dt} = \nabla \cdot S + \rho b \tag{2}$$

Where, ρ = Density, V = Velocity vector, S = Cauchy stress tensor, b = Specific body force vector.

For Williamson fluid Cauchy stress tensor is given by equation,

$$S = -pI + \tau \tag{3}$$

$$\tau = \left[\mu_\infty + \frac{(\mu_0 - \mu_\infty)}{1 - \Gamma\gamma}\right] A_1 \tag{4}$$

Where, p = Pressure, I = Identity vector, τ = Extra stress tensor, μ0 =Limiting viscosities at zero, μ∞ = Limiting viscosities at infinite, Γ = Time Constant, A1 = The First Rivlin- Ericksen tensor, And γ is defined by the below equation

$$\gamma = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \tag{5}$$

In equation (5), π denotes the second invariant strain tensor and it is given by the equation,

$$\pi = trace(A_1^{\,2}) \tag{6}$$

Now, The First Rivlin-Ericksen tensor A_1 is given by,

$$(\mathrm{Aij})_1 = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \qquad 1 \le i, j \le n$$

$$\therefore\ A_1 = \begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} \end{pmatrix} \tag{7}$$

From equations (6.5)-(6.7) we get,

$$\therefore\ \gamma = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]^{\frac{1}{2}} \tag{8}$$

Here we considered only the case limiting viscosity at infinite is zero and $\Gamma\gamma < 1$

So, the extra stress tensor takes the form

$$\tau = \left[\frac{\mu_0}{1 - \Gamma\gamma}\right] A_1 \tag{9}$$

By taking binomial expansion and second and higher order terms neglected, We get,

$$\tau = \mu_0 \left[1 + \Gamma\gamma\right] A_1 \tag{10}$$

The extra stress tensor components are

$$\tau_{xx} = 2\mu_0 \left[1 + \Gamma\gamma\right] \frac{\partial u}{\partial x} \tag{11}$$

$$\tau_{xy} = \tau_{yx} = 2\mu_0 \left[1 + \Gamma\gamma\right] \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{12}$$

$$\tau_{yy} = 2\mu_0 \left[1 + \Gamma\gamma\right] \frac{\partial v}{\partial y} \tag{13}$$

$$\tau_{xz} = \tau_{yz} = \tau_{zx} = \tau_{zy} = \tau_{zz} = 0 \tag{14}$$

Let us consider two–dimensional, steady an incompressible flow of Williamson fluid over a semi infinite moving surface keeping the origin fixed in the absence of body forces. Due to the linear stretching flow is generated.The governing equations of above defined flow problem in the direction of flow (x-axis) and normal to the flow (y-axis), respectively

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \tag{15}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y} \tag{16}$$

For Williamson fluid model it is given by substitute the values of equation (8) in equation (11)-(12).

$$\tau_{xx}=2\mu_0\left\{1+\Gamma\left[\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]^{\frac{1}{2}}\right\}\times\frac{\partial u}{\partial x} \tag{17}$$

$$\tau_{xy}=\tau_{yx}=\mu_0\left\{1+\Gamma\left[\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]^{\frac{1}{2}}\right\}\times\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \tag{18}$$

$$\tau_{yy}=2\mu_0\left\{1+\Gamma\left[\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]^{\frac{1}{2}}\right\}\times\frac{\partial v}{\partial y} \tag{19}$$

In the absence of a pressure gradient P and using equations (17)-(19) equations (15) and (16) are converted into,

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+\sqrt{2}\,\nu\Gamma\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} \tag{20}$$

Under the boundary conditions are:

$$u=Bx=U_\infty \quad and\ v=0\quad at\ y=0 \tag{20a}$$

$$u=0\quad at\ y\to\infty \tag{20b}$$

Where, ν = kinematic viscosity, U_∞= velocity at the wall, B > 0 stretching parameter along X-axis.

3. Derivation of Similarity Equations and Numerical Solution

Applying similarity technique the relation between dependent variables and independent variable are as under:

$$u=Bxf'(\eta)\ ,\quad v=-\sqrt{B\nu}\,f(\eta)\ ,\quad \eta=\sqrt{\frac{B}{\nu}}y \tag{21}$$

Substitute transformations equations (21) in equation (20) we get non linear ordinary differential equation as,

$$f'''(\eta)-f'^2(\eta)+f(\eta)f'(\eta)+\lambda f''(\eta)f'''(\eta)=0 \tag{22}$$

With the corresponding boundary conditions:

$$f(0)=0, f'(0)=1 \tag{23}$$

$$f'(\infty)=0 \tag{24}$$

Where, $\lambda=\Gamma X\sqrt{\frac{2B^2}{\nu}}$ = the dimensionless Williamson parameter

We convert equation (22) into linear ordinary differential equation by quasi-linearization method as,

$$\begin{aligned}&f_n'''-f_n'f_n'-f_nf_n''+\lambda f_n''f_n'''+f_n''[f_{n+1}-f_n]\\&-2f_n'[f_{n+1}'-f_n']+f_n[f_{n+1}''-f_n'']\\&+\lambda f_n'''[f_{n+1}''-f_n'']+f_{n+1}'''-f_n'''+\lambda f_n''[f_{n+1}'''-f_n''']=0\end{aligned} \tag{25}$$

Simplifying above equation we get,

$$\begin{aligned}&f_n'f_n'+f_n''f_{n+1}-2f_n'f_{n+1}'+f_nf_{n+1}''-f_nf_n''\\&+\lambda f_n'''f_n''+f_{n+1}'''+\lambda f_n''f_{n+1}'''-\lambda f_n''f_n'''=0\end{aligned} \tag{26}$$

To fit the curve, consider the solution

$$f_n=A\eta^2+B\eta+C \tag{27}$$

Using the boundary conditions given by equations (23) and (24),

$$A=-0.5, B=1, C=0$$

So, equation (27) can be written as,

$$f_n=-0.5\eta^2+\eta$$

$$\therefore f_n'=-\eta+1$$

$$\therefore f_n''=-1$$

$$\therefore f_n'''=0$$

Substitute all above the values in equation (26),

$$(1-\lambda)f'''_{n+1} + (-0.5\eta^2+\eta)f''_{n+1} + (2\eta-2)f'_{n+1} - f_{n+1} = -0.5\eta^2+\eta-1 \quad (28)$$

4. Numerical Solution: Finite Difference Method (FDM)

Here the equation (28) is linear ordinary differential equation now we will find its numerical solution using Finite difference method,

Now central difference formulas for derivatives are:

$$f_i'(\eta) = \frac{f_{i+1}-f_{i-1}}{2h} + O(h) \quad (29)$$

$$f_i''(\eta) = \frac{f_{i-1}-2f_i+f_{i+1}}{h^2} + O(h^2) \quad (30)$$

$$f_i'''(\eta) = \frac{f_{i+2}-2f_{i+1}+2f_{i-1}-f_{i-2}}{2h^3} + O(h^3) \quad (31)$$

Substitute the values of equations (29) – (31) in equation (28) at the i[th] node we get the system of linear equations,

$$\begin{aligned}(1-\lambda)f_{i+2} &+ (-2+2\lambda-h\eta^2+2h\eta+2h^2\eta-2h^2)f_{i+1} \\ &+(2h\eta^2-4h\eta+2h^3)f_i \\ &+(2-2\lambda-h\eta^2+2h\eta-2h^2\eta+2h^2)f_{i-1} \\ &+(\lambda-1)f_{i-2} = (-1+\eta-0.5\eta^2)2h^3\end{aligned} \quad (32)$$

Which is system of linear equations we solve it in MATLAB by dividing [0,1] interval into 500 subinterval having length h=0.002.solution is presented graphically as below.

5. Graphical Presentation

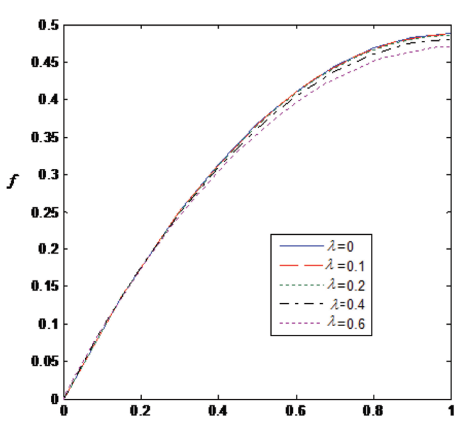

Figure 1: Solution graph.

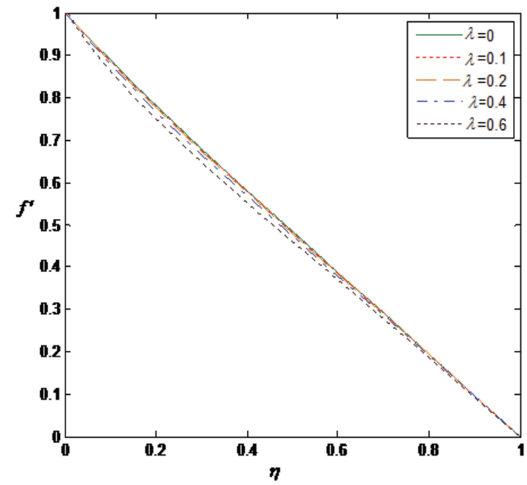

Figure 2: Velocity graph.

6. Comparision of Results

Using Similarity group transformation method, the PDE's of flow problems are transformed into ODE's. The ODE's are then solved numerically using Finite difference method and presented graphically. Figure 1 shows that f increase with an increase in Williamson fluid parameter λ. Figure 2 represents the velocity profiles f' for different values of Williamson fluid parameter λ.

Nadeem et al. (2013) studied the two-dimensional flow of Williamson fluid. They applied similarity techniques on the governing flow equations. Then that equation are solved analytically using the homotopy analysis method. They discussed the effect of Williamson parameter on solution function and velocity profile.

The graphs of solution function and velocity profiles are almost match with the graphs given by Nadeem et al. (2013). Therefore the numerical solution discussed in this paper are very similar to the analytical solutios obtained by Nadeem et al.

Conclusion

The governing equations of motion (partial differential equations) of the Williamson boundary layer flow of non-Newtonian fluid are solved numerically using FDM. It shows that velocity field and boundary layer thickness decrease with increase the values of λ.

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