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Edge Vertex Prime Labeling for $K_{2,n}$ and $K_{3,n}$ Graphs

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Abstract In my study, I inspect edge vertex prime labeling of some graphs like complete bipartite graphs $K_{2,n}$ & $K_{3,n}$. I proved that the graphs $K_{2,n}$ for every n & $K_{3,n}$ for $n = 3, 4, ..., 29$ are edge vertex prime.

Keywords: Complete bipartite graph; edge vertex prime labeling; graph labeling; prime labeling.

1. INTRODUCTION

In my study, all graphs $G = (V, G, E(G))$ are finite, directionless and simple graphs. $V(G)$ will symbolized the vertex set and $E(G)$ will symbolized edge set of the graph. $|V(G)|$ and $|E(G)|$ will symbolized the number of vertices and number of edges of graph one by one. For various graphs, theorical symbolization and vocabulary, I survey Gross and Yeelen [3] and for Number theory, I survey Burton [1].

Definition 1.1. *Graph labeling* is an assignment of integers either to the vertices or edges or both subject to certain conditions.

A dynamic survey on graph labeling is regularly updated by Gallian [2].

Definition 1.2. *Prime labeling* is a function $f: V(G) \to \{1, 2, \ldots, n\}$ with one-to-one correspondence and each edge $e = uv$, $\gcd(f(u), f(v)) = 1$.

Prime graph is a graph which has prime labeling.

Prime labeling notation was invented by Roger Entringer and familiarized in an article by Taut, Dabboucy and Howalla [6].

Definition 1.3.*Edge vertex prime labeling* is a function $f: V(G) \cup E(G) \to \{1, 2, \ldots, |V(G) \cup E(G)|\}$ with one-to-one correspondence and for any edge $e = uv$; $f(u), f(v)$ and $f(uv)$ are pairwise relatively prime.

Edge vertex prime graph is a graph, which admits edge vertex prime labeling.

R. Jagadesh and J. Baskar Babujee [4] introduced an edge vertex prime labeling. They proved the existence of paths, cycles and star-$K_{1,n}$.Y Parmar [5] proved that

- The wheel graph W_n is an edge vertex prime graph for every n.
- The fan graph f_n is an edge vertex prime graph for every n.
- The friendship graph F_n is an edge vertex prime graph for every n.

Definition 1.4. A simple bipartite graph G, with bipartition $V = S \cup T$, every vertex in S is joined to every vertex of T is called complete bipartite graph. If S has n' vertices and T has n vertices, such a graph is denoted by $K_{n',n}$.

2. MAIN THEOREMS

Theorem 2.1 *The complete bipartite graph $K_{2,n}$ is an edge vertex prime graph.*

Proof: Let $G = K_{2,n}$ is a complete bipartite graph and

$$V(G) = \{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_n\}$$

$$E(G) = \{u_1 v_j \,/\, 1 \le j \le n\} \cup \{u_2 v_j \,/\, 1 \le j \le n\}$$

Order of complete bipartite graph $K_{2,n}$ is, $|V(G)| = n+2$ and size is, $|E(G)| = 2n$. Thus we have $|E(G)| + |V(G)| = 3n+2$.

Assume that is a largest prime number occur in the set $\{1, 2, \ldots, 3n+2\}$.

Let $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G) \cup E(G)|\}$ is a function with one-to-one correspondence and it is defined as follows:

<u>Case -1</u>: $p = 3n + 2$

$$f(u_1) = 1$$

$$f(u_2) = 3n + 2 = p$$

$$f(v_j) = 3j; \forall j$$

$$f(u_1 v_j) = 3j - 1; \forall j$$

$$f(u_2 v_j) = 3j + 1; \forall j$$

Now our claims are

(1) $f(u_1)$, $f(v_j)$ and $f(u_1 v_j)$ are relatively prime in pairwise.

(2) $f(u_2)$, $f(v_j)$ and $f(u_2 v_j)$ are relatively prime in pairwise.

(1) $\gcd\left(f\left(u_1\right), f\left(v_j\right)\right) = \gcd\left(1, 3j\right) = 1, \forall j$

$\gcd\left(f\left(u_1\right), f\left(u_1 v_j\right)\right) = \gcd\left(1, 3j-1\right) = 1, \forall j$

$\gcd\left(f\left(v_j\right), f\left(u_1 v_j\right)\right) = \gcd\left(3j, 3j-1\right) = 1, \forall j$, since $3j$ and $3j - 1$ are conjugative numbers.

(2) $\gcd\left(f\left(u_2\right), f\left(v_j\right)\right) = \gcd\left(p, 3j\right) = 1, \forall j$, since p is the largest prime in the corresponding set and there are no such products in the set, which has p as a multipier.

$\gcd\left(f\left(u_2\right), f\left(u_2 v_j\right)\right) = \gcd\left(p, 3j+1\right) = 1, \forall j$

$\gcd\left(f\left(v_j\right), f\left(u_2 v_j\right)\right) = \gcd\left(3j, 3j+1\right) = 1, \forall j$, since $3j$ and $3j + 1$ are conjugative numbers.

Thus, for $p = 3n + 2, K_{2,n}$ is an edge vertex prime graph.

Case -2: $p = 3n - (3t+1); t = 0, 1, \ldots, N$ (natural numbers)

$f\left(u_1\right) = 1$

$f\left(u_2\right) = p$

$$f\left(v_i\right) = \begin{cases} 3j & ; \quad j = 1, 2, \ldots, n-(t+1) \\ 3j+1 & ; \quad j = n-t, n-t+1, \ldots, n \end{cases}$$

$$f\left(u_1 v_i\right) = \begin{cases} 3j-1 & ; \quad j = 1, 2, \ldots, n-(t+1) \\ 3j & ; \quad j = n-t, n-t+1, \ldots, n \end{cases}$$

$$f\left(u_2 v_j\right) = \begin{cases} 3j+1 & ; \quad j = 1, 2, \ldots, n-(t+1) \\ 3j+2 & ; \quad j = n-t, n-t+1, \ldots, n \end{cases}$$

Now our claims are

(1) $f\left(u_1\right), f\left(v_j\right)$ and $f\left(u_1 v_j\right)$ are relatively prime in pairwise.

(2) $f\left(u_2\right), f\left(v_j\right)$ and $f\left(u_2 v_j\right)$ are relatively prime in pairwise.

(1) $$\gcd\left(f\left(u_1\right), f\left(v_j\right)\right) = \begin{cases} \gcd\left(1, 3j\right) & ; \quad j = 1, 2, \ldots, n-(t+1) \\ \gcd\left(1, 3j+1\right) & ; \quad j = n-t, n-t+1, \ldots, n \end{cases}$$
$$= 1$$

$$\gcd\left(f\left(u_1\right),f\left(u_1v_j\right)\right)=\begin{cases}\gcd\left(1,3j-1\right) & ; \quad j=1,2,\ldots,n-\left(t+1\right)\\ \gcd\left(1,3j\right) & ; \quad j=n-t,n-t+1,\ldots,n\end{cases}$$
$$=1$$

$$\gcd\left(f\left(v_j\right),f\left(u_1v_j\right)\right)=\begin{cases}\gcd\left(3j,3j-1\right) & ; \quad j=1,2,\ldots,n-\left(t+1\right)\\ \gcd\left(3j+1,3j\right) & ; \quad j=n-t,n-t+1,\ldots,n\end{cases}$$
$$=1$$

Since $3j-1$, $3j$ and $3j+1$ are conjugative numbers.

(2) $$\gcd\left(f\left(u_2\right),f\left(v_j\right)\right)=\begin{cases}\gcd\left(p,3j\right) & ; \quad j=1,2,\ldots,n-\left(t+1\right)\\ \gcd\left(p,3j+1\right) & ; \quad j=n-t,n-t+1,\ldots,n\end{cases}$$
$$=1$$

$$\gcd\left(f\left(u_2\right),f\left(u_2v_j\right)\right)=\begin{cases}\gcd\left(p,3j+1\right) & ; \quad j=1,2,\ldots,n-\left(t+1\right)\\ \gcd\left(p,3j+2\right) & ; \quad j=n-t,n-t+1,\ldots,n\end{cases}$$
$$=1$$

$$\gcd\left(f\left(v_j\right),f\left(u_2v_j\right)\right)=\begin{cases}\gcd\left(3j,3j+1\right) & ; \quad j=1,2,\ldots,n-\left(t+1\right)\\ \gcd\left(3j+1,3j+2\right) & ; \quad j=n-t,n-t+1,\ldots,n\end{cases}$$
$$=1$$

Thus, for $p=3n-\left(3t+1\right);K_{2,n}$ is an edge vertex prime graph.

<u>Case -3</u>: $p=3n-\left(3t-1\right);t=0,1,\ldots,N$ (natural numbers)

$f\left(u_1\right)=1$

$f\left(u_2\right)=p$

$$f\left(v_j\right)=\begin{cases}3j & ; \quad j=1,2,\ldots,n-\left(t+1\right)\\ 3j-1 & ; \quad j=n-t\\ 3j+1 & ; \quad j=n-t+1,\ldots,n\end{cases}$$

$$f\left(u_1v_j\right)=\begin{cases}3j-1 & ; \quad j=1,2,\ldots,n-\left(t+1\right)\\ 3j & ; \quad j=n-t,n-t+1,\ldots,n\end{cases}$$

$$f(u_2v_j)=\begin{cases}3j+1 & ; \quad j=1,2,\ldots,n-(t+1)\\ 3j+2 & ; \quad j=n-t,n-t+1,\ldots,n\end{cases}$$

Now our claims are

(1) $f(u_1), f(v_j)$ and $f(u_1v_j)$ are relatively prime in pairwise.

(2) $f(u_2), f(v_j)$ and $f(u_2v_j)$ are relatively prime in pairwise.

(1) $$\gcd(f(u_1), f(v_j))=\begin{cases}\gcd(1,3j) & ; \quad j=1,2,\ldots,n-(t+1)\\ \gcd(1,3j-1) & ; \quad j=n-t\\ \gcd(1,3j+1) & ; \quad j=n-t+1,\ldots,n\end{cases}$$
$$=1$$

$$\gcd(f(u_1), f(u_1v_j))=\begin{cases}\gcd(1,3j-1) & ; \quad j=1,2,\ldots,n-(t+1)\\ \gcd(1,3j) & ; \quad j=n-t,n-t+1,\ldots,n\end{cases}$$
$$=1$$

$$\gcd(f(v_j), f(u_1v_j))=\begin{cases}\gcd(3j,3j-1) & ; \quad j=1,2,\ldots,n-(t+1)\\ \gcd(3j-1,3j) & ; \quad j=n-t\\ \gcd(3j+1,3j) & ; \quad j=n-t+1,\ldots,n\end{cases}$$
$$=1$$

Since $3j-1$, $3j$ and $3j+1$ are conjugative numbers.

(2) $$\gcd(f(u_2), f(v_j))=\begin{cases}\gcd(p,3j) & ; \quad j=1,2,\ldots,n-(t+1)\\ \gcd(p,3j-1) & ; \quad j=n-t\\ \gcd(p,3j+1) & ; \quad j=n-t+1,\ldots,n\end{cases}$$
$$=1$$

$$\gcd(f(u_2), f(u_2v_j))=\begin{cases}\gcd(p,3j+1) & ; \quad j=1,2,\ldots,n-(t+1)\\ \gcd(p,3j+2) & ; \quad j=n-t,n-t+1,\ldots,n\end{cases}$$
$$=1$$

$$\gcd\left(f\left(v_j\right), f\left(u_2 v_j\right)\right) = \begin{cases} \gcd\left(3j, 3j+1\right) & ; \quad j = 1,2,\ldots,n-\left(t+1\right) \\ \gcd\left(3j-1, 3j+2\right) & ; \quad j = n-t \\ \gcd\left(3j+1, 3j+2\right) & ; \quad j = n-t+1,\ldots,n \end{cases}$$
$$= 1$$

Thus, for $p = 3n - \left(3t-1\right); K_{2,n}$ is an edge vertex prime graph.
Hence, complete bipartite graph $K_{2,n}$ is an edge vertex prime graph.
Illustration 2.1 for case-1: The complete bipartite graph $G = K_{2,9}$ is an edge vertex prime graph, which is shown in the Figure 1.

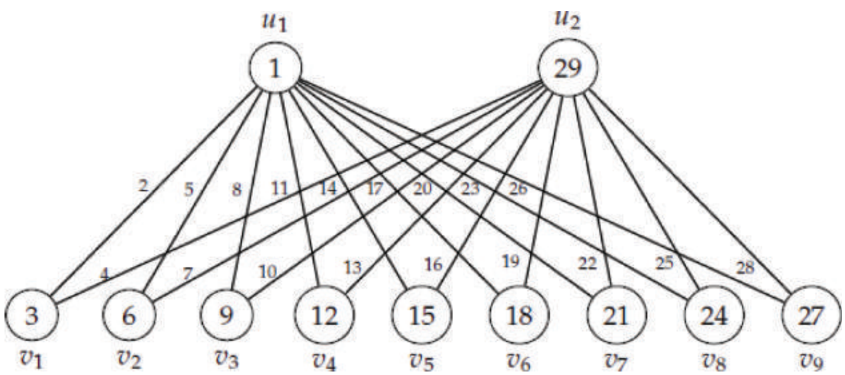

Figure 1: $K_{2,9}$

Illustration 2.2 for case-2: The complete bipartite graph $G = K_{2,8}$ is an edge vertex prime graph with $t = 0$, which is shown in the Figure 2.

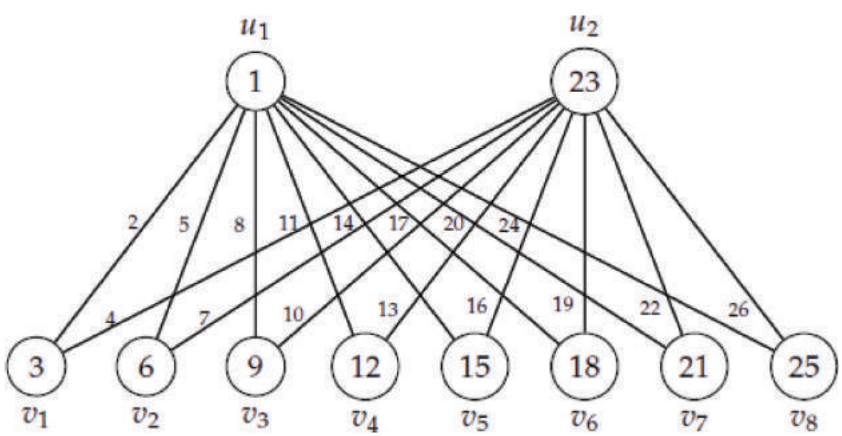

Figure 2: $K_{2,8}$

For $K_{2,8}, |E(G)|+|V(G)| = 3n+2 = 26$. So the largest prime number in the set$\{1, 2, \ldots, 26\}$ is $p = 23$.
i.e. $p = 3(8)-1 = 3n-1$.
Here we choose case-2, $p = 3n - (3t + 1)$ and compare it to $3n - 1$, we get $t = 0$. Since in case-3, $p = 3n - (3t - 1)$, we get $t = 2/3$, which is not possible because t is natural number with 0.

Illustration 2.3 for case-3: The complete bipartite graph $G = K_{2,10}$ is an edge vertex prime graph with $t = 0$, which is shown in the Figure 3.

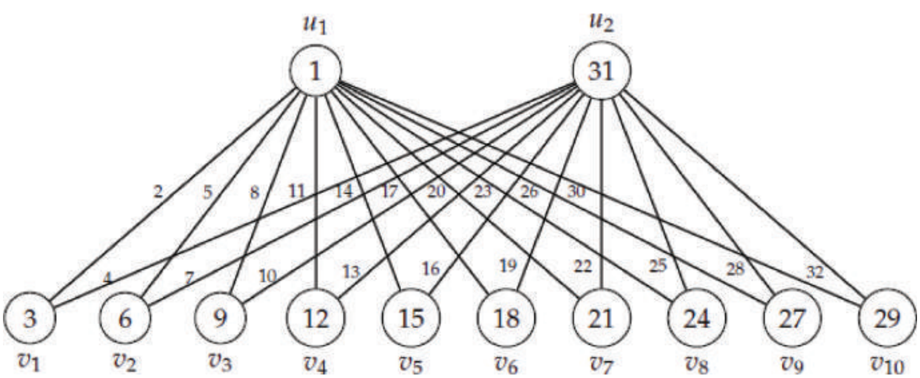

Figure 3: $K_{2,10}$

Illustration 2.4 for case-3: The complete bipartite graph $G = K_{2,11}$ is an edge vertex prime graph with $t = 1$, which is shown in the Figure 4.

Theorem 2.2 *The complete bipartite graph* $K_{3,n}; n = 3, 4, \ldots, 29$ *is an edge vertex prime graph.*

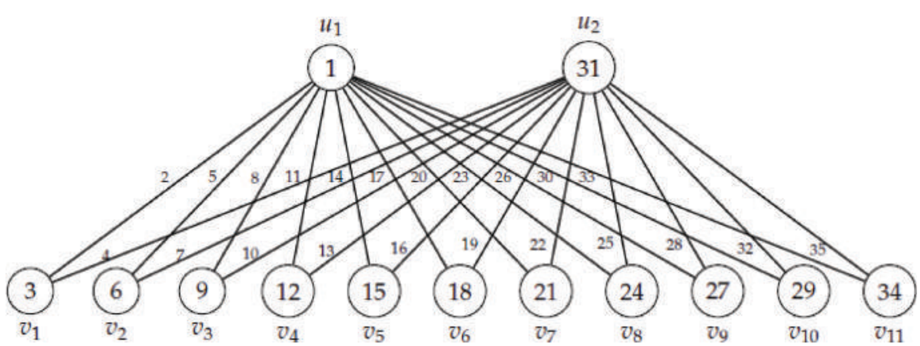

Figure 4: $K_{2,11}$

Proof: Let $G = K_{3,n}; n = 3,4,\ldots,29$ is a complete bipartite graph and

$V(G) = \{u_1, u_2, u_3\} \cup \{v_1, v_2, \ldots, v_n\}$

$E(G) = \{u_1v_j / 1 \le j \le n\} \cup \{u_2v_j / 1 \le j \le n\} \cup \{u_3v_j / 1 \le j \le n\}$

Order of complete bipartite graph $K_{2,n}$ is, $|V(G)| = n+3$ and size is, $|E(G)| = 3n$.
Thus we have $|E(G)| + |V(G)| = 4n+3$.

Let $f : V(G) \cup E(G) \to \{1,2,\ldots,|V(G) \cup E(G)|\}$ is a function with one-to-one correspondence and it is defined as follows:

Case -1: $n = 3$

$f(u_1) = 1$

$f(u_2) = 11$

$f(u_3) = 13$

$f(v_1) = 3$

$f(v_2) = 5$

$f(v_3) = 7$

$f(u_1v_j) = 4j - 2; j = 1,2,3$

$f(u_2v_j) = 4j; j = 1,2,3$

$$f(u_2v_j) = \begin{cases} 14j & ; \quad j = 1 \\ 4j+1 & ; \quad j = 2 \\ 4j+3 & ; \quad j = 3 \end{cases}$$

Here $f(u_1), f(u_2), f(u_3), f(v_1), f(v_2)$ and $f(v_3)$ are prime numbers and $f(u_1v_j), f(u_2v_j)$ and $f(u_3v_j)$ are not multiple of $f(u_1), f(v_j); f(u_2), f(v_j)$ and $f(u_3), f(v_j)$ respectively.
So $f(u_1), f(v_j), f(u_1v_j); f(u_2), f(v_j), f(u_2v_j)$ and $f(u_3), f(v_j), f(u_3v_j)$ are relatively prime in pairwise.
Thus, $K_{3,3}$ is an edge vertex prime graph.

Case -2: n = 6,8,9,12,13, …, 20

$f(u_1) = 1$

$f(u_2) =$ second largest prime in $4n + 3$ numbers.

$f(u_3) =$ largest prime in $4n + 3$ numbers.

$f(v_j) =$ remaining all primes in ascending order except 2.

$$f(u_1v_j) = \begin{cases} 4j-2 & ; \quad j = 1,2,3,4 \\ 4j & ; \quad j = 5,6,\ldots,20 \end{cases}$$

$$f(u_2v_j) = \begin{cases} 4j & ; \quad j = 1 \\ 4j+1 & ; \quad j = 2 \\ 4j+3 & ; \quad j = 3 \\ 4j+2 & ; \quad j = 4,5,\ldots,20 \end{cases}$$

$$f(u_3v_j) = \begin{cases} 4j+4 & ; \quad j = 1,2,3 \\ 4j+5 & ; \quad j = 4,5,7,10,11 \\ 4j+3 & ; \quad j = 6,8,9,12,13,15,18 \\ 4j+1 & ; \quad j = 14,16,17,19,20 \end{cases}$$

Now, the greatest common divisor of $(u_1), f(u_1v_j); f(u_2), f(u_2v_j); f(u_3), f(u_3v_j)$; $f(u_1), f(v_j); f(u_2), f(v_j); f(u_3), f(v_j); f(v_j), f(u_1v_j)$; $f(v_j), f(u_2v_j); f(v_j), f(u_3v_j)$ is one.

i.e, $f(u_1), f(v_j), f(u_1v_j); f(u_2), f(v_j), f(u_2v_j)$ and $f(u_3), f(v_j), f(u_3v_j)$ are relatively prime in pairwise.

Thus, $K_{3,n}; n = 6,8,9,12,13,\ldots,20$ is an edge vertex prime graph.

Case -3: $n = 4,5,7,10,11$

$f(u_1) = 1$

$f(u_2) =$ second largest prime in $4n + 3$ numbers.

$f(u_3) =$ largest prime in $4n + 3$ numbers.

$f(v_j) =$ remaining all primes in ascending order except 2 and 3.

$f(u_1v_j) = 4j; j = 1,2,\ldots,11$

$$f(u_2v_j) = \begin{cases} 4j-2 & ; \quad j = 1 \\ 4j-5 & ; \quad j = 2 \\ 4j-3 & ; \quad j = 3 \\ 4j-1 & ; \quad j = 4,7,9,10 \\ 4j+1 & ; \quad j = 5,6,8,11 \end{cases}$$

$f(u_3v_j) = 4j+2; j = 1,2,\ldots,11$

Here $f(u_1), f(v_j)$ & $f(u_1v_j); f(u_2), f(v_j)$ & $f(u_2v_j)$ and $f(u_3), f(v_j)$ & $f(u_3v_j)$ are relatively prime in pairwise.

Thus, $K_{3,n}; n = 4,5,7,10,11$ is an edge vertex prime graph.

Case -4: $n = 21,22,25,26,27,28$

$f(u_1) = 1$

$f(u_2) =$ second largest prime in $4n+3$ numbers.

$f(u_3) =$ largest prime in $4n+3$ numbers.

$f(v_j) =$ remaining all primes in ascending order.

$$f(u_1v_j) = \begin{cases} 4j+5 & ; \quad j=1 \\ 2j & ; \quad j=2,3,4 \\ 4j+4 & ; \quad j=5 \\ 4j-2 & ; \quad j=6 \\ 4j+2 & ; \quad j=7,8,\ldots,28 \end{cases}$$

$$f(u_2v_j) = \begin{cases} 4j+11 & ; \quad j=1 \\ 4j+2 & ; \quad j=2,4 \\ 4j & ; \quad j=3,24,25 \\ 4j+5 & ; \quad j=5,7,10,11 \\ 4j+3 & ; \quad j=6,8,9,12,13,15,18,22,27,28 \\ 4j+1 & ; \quad j=14,16,17,19,20,21,23,26 \end{cases}$$

$$f(u_3v_j) = \begin{cases} 4j+17 & ; \quad j=1 \\ 4j+6 & ; \quad j=2,5 \\ 4j+4 & ; \quad j=3,4,6,8,9,20 \\ 4j+5 & ; \quad j=7 \\ 4j+3 & ; \quad j=21 \\ 4j & ; \quad j=22,23,26,27,28 \\ 4j-1 & ; \quad j=24,25 \end{cases}$$

Here $f(u_1), f(v_j)$ & $f(u_1v_j); f(u_2), f(v_j)$ & $f(u_2v_j)$ and $f(u_3), f(v_j)$ & $f(u_3v_j)$ are relatively prime in pairwise.

Thus, $K_{3,n}; n = 21,22,25,26,27,28$ is an edge vertex prime graph.

Case -5: $n = 23, 24, 29$

$f(u_1) = 1$

$f(u_2) =$ second largest prime in $4n + 3$ numbers.

$f(u_3) =$ largest prime in $4n + 3$ numbers.

$f(v_j) =$ remaining all primes with number 4 in ascending order.

$$f(u_1 v_j) = \begin{cases} 4j+5 & ; \quad j=1 \\ 4j & ; \quad j=2 \\ 8j+1 & ; \quad j=3 \\ 4j+2 & ; \quad j=4,5,6,7,11,12,\ldots,29 \\ 4j+3 & ; \quad j=8 \\ 4j+6 & ; \quad j=9 \\ 4j-2 & ; \quad j=10 \end{cases}$$

$$f(u_2 v_j) = \begin{cases} 15j & ; \quad j=1 \\ 5j & ; \quad j=2 \\ 9j & ; \quad j=3 \\ 2j-2 & ; \quad j=4 \\ 4j-4 & ; \quad j=5 \\ 4j & ; \quad j=6,7,\ldots,29 \end{cases}$$

$$f(u_3 v_j) = \begin{cases} 21j & ; \quad j=1 \\ 7j & ; \quad j=2 \\ 11j & ; \quad j=3 \\ 3j & ; \quad j=4 \\ 4j & ; \quad j=5 \\ 4j+15 & ; \quad j=6 \\ 4j+6 & ; \quad j=7 \\ 4j+13 & ; \quad j=8,9 \\ 4j+11 & ; \quad j=10,11,13,16 \\ 4j+9 & ; \quad j=12,14,15,17,18,19 \\ 4j+7 & ; \quad j=20,21,26,27 \\ 4j+5 & ; \quad j=22,25,28 \\ 4j+3 & ; \quad j=23,24,29 \end{cases}$$

Here $f(u_1), f(v_j)$ & $f(u_1 v_j); f(u_2), f(v_j)$ & $f(u_2 v_j)$ and $f(u_3), f(v_j)$ & $f(u_3 v_j)$ are relatively prime in pairwise.

Thus, $K_{3,n}; n = 23, 24, 29$ is an edge vertex prime graph.

Illustration 2.5 for case-1: The complete bipartite graph $G = K_{3,3}$ is an edge vertex prime graph, which is shown in the Figure 5.

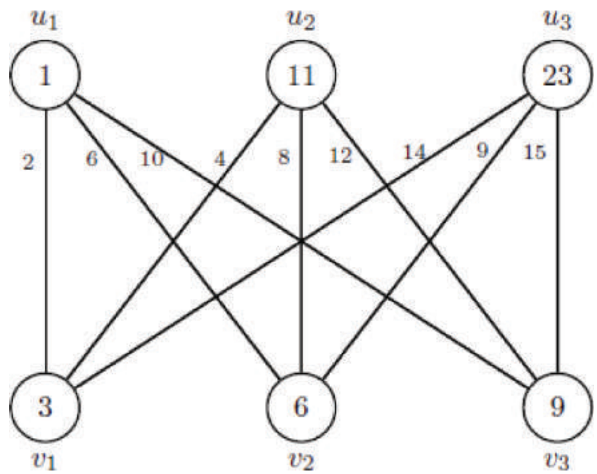

Figure 5: $K_{3,3}$

Illustration 2.6 for case-2: The complete bipartite graph $G = K_{3,14}$ is an edge vertex prime graph, which is shown in the Figure 6.

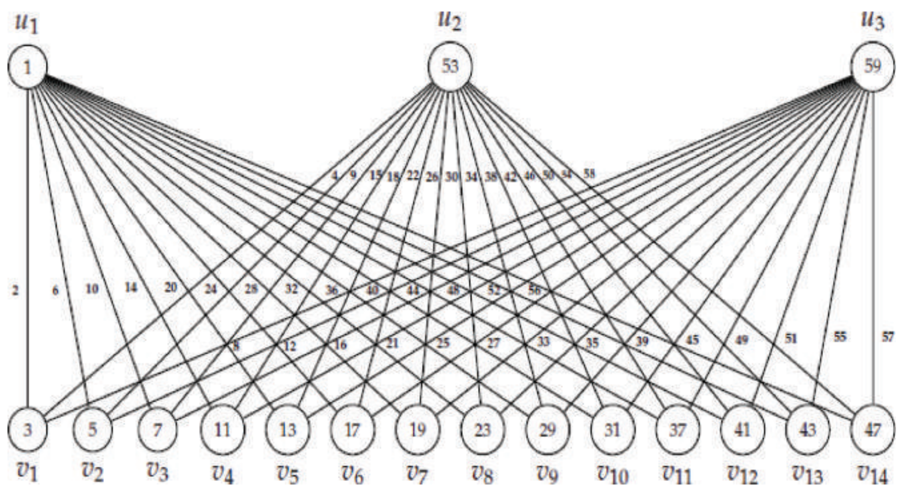

Figure 6: $K_{3,14}$

Illustration 2.7 for case-3: The complete bipartite graph $G = K_{3,5}$ is an edge vertex prime graph, which is shown in the Figure 7.

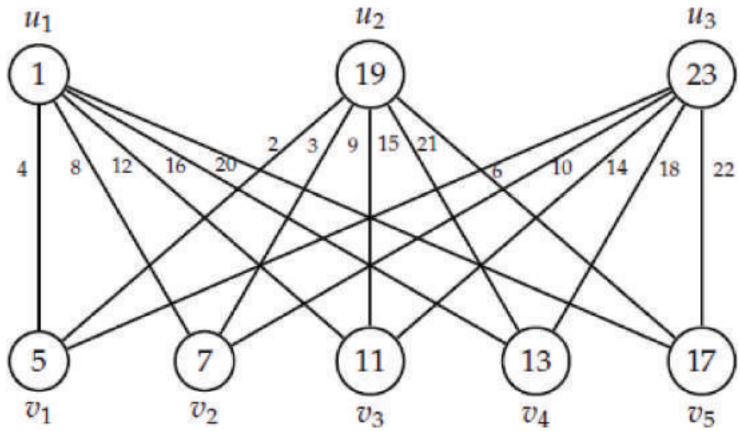

Figure 7: $K_{3,5}$

Illustration 2.8 for case-4: The complete bipartite graph $G = K_{3,25}$ is an edge vertex prime graph, which is shown in the Figure 8.

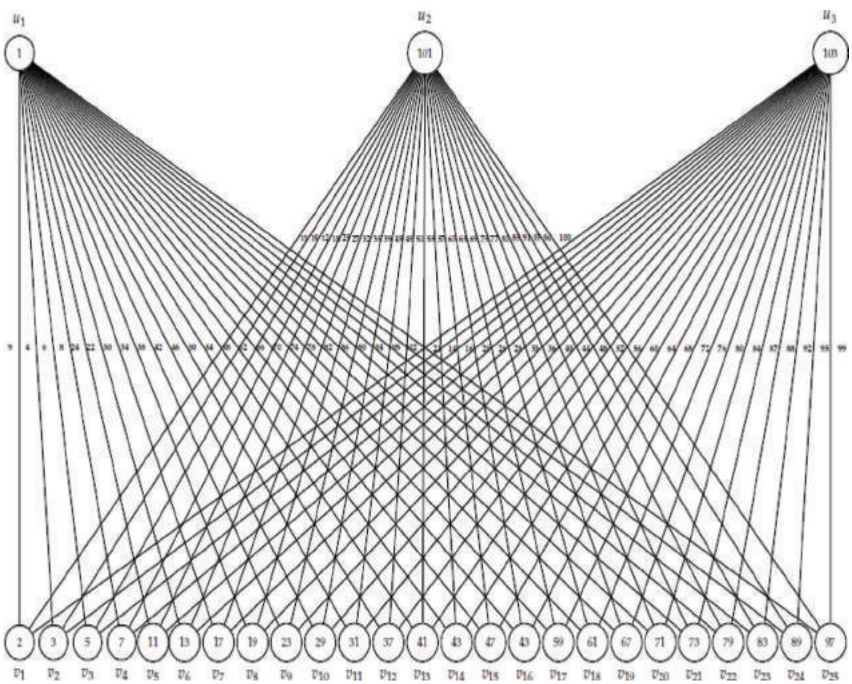

Figure 8: $K_{3,25}$

Illustration 2.9 for case-5: The complete bipartite graph $G = K_{3,23}$ is an edge vertex prime graph, which is shown in the Figure 9.

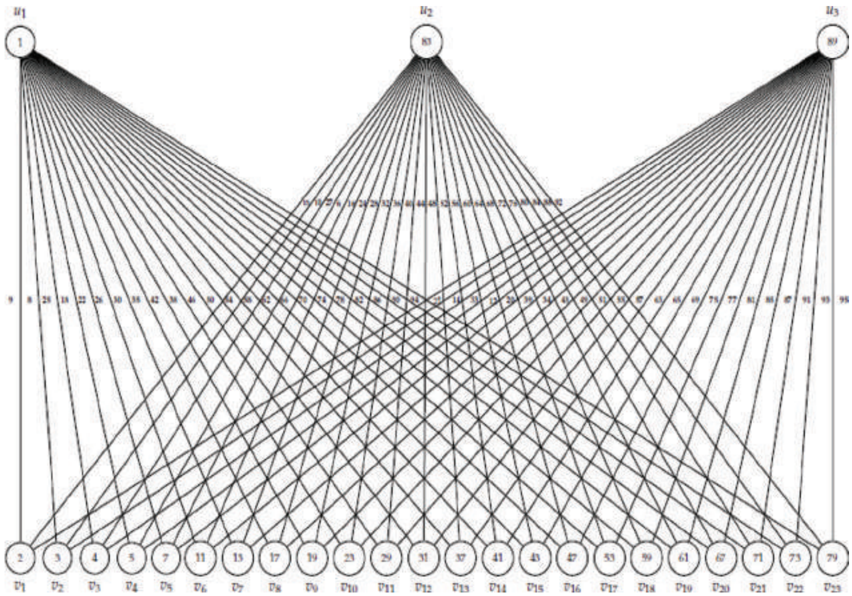

Figure 9: $K_{3,23}$

CONCLUSION

Here, we have derived two new results related to edge vertex prime labeling technique. To explore some new edge vertex prime graphs is an open problem.

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