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On Color Energy of Few Classes of Bipartite Graphs and Corresponding Color Complements

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ABSTRACT

For a given colored graph G, the color energy is defined as $E_c(G) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i is a color eigenvalue of the color matrix of G, $A_c(G)$ with entries as 1, if both the corresponding vertices are neighbors and have different colors; -1, if both the corresponding vertices are not neighbors and have same colors and 0, otherwise. In this article, we study color energy of graphs with proper coloring and $L(h, k)$-coloring. Further, we examine the relation between $E_c(G)$ with the color energy of the color complement of a given graph G and other graph parameters such as chromatic number and domination number.

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1. Introduction

We consider throughout this paper only undirected graphs with order n and size m, which are simple and finite. For related concepts and definitions, we refer to (Chartrand and Zhang 2008, Cvetkovic, Doob and Sachs 1980, West 2001) and for notations not given explicitly we refer to (Adiga, Sampathkumar, Sriraj and Shrikanth 2013, Joshi and Joseph 2017a, Joshi and Joseph 2017b). For a graph G, its energy $E(G) = \sum_{j=1}^{n} |\lambda_j|$, where lj is an eigenvalue of the adjacency matrix $A(G)$ (Gutman 1978, Gutman and Zhang 2012). Gutman was the one who initiated the concept of energy for an arbitrary graph in 1978 (Gutman 1978). He was motivated by the theory in chemistry that describes *energy* of a molecular graph as the total energy of π-electrons (Gutman and Zhang 2012). Molecular graph is a representation of a molecular compound where in carbon atoms represent vertices and bonds between them represent edges. Due to the significance of graph energy in chemistry this topic has attracted several researchers. Many new variations of $E(G)$ like Laplacian energy, distance energy etc. have contributed in the enhancement of the theory of graph energy (Gutman and Zhang 2012, Gutman and Zhou 2006, Indulal, Gutman and Ambat 2010).

In 2013, Adiga et al. (Adiga, Sampathkumar, Sriraj and Shrikanth 2013) initiated the study of color matrix $A_c(G) = [a_{ij}]_{n\times n}$ where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent vertices in } G \text{ and } c(v_i) \neq c(v_j) \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non adjacent vertices in } G \text{ and } c(v_i) = c(v_j) \\ 0, & \text{otherwise} \end{cases}$$

$A_c(G)$ is a special case of L-matrix which was defined by Sampathkumar and Sriraj in (Sampathkumar and Sriraj 2013). Adiga et al. (Adiga, Sampathkumar, Sriraj and Shrikanth 2013) defined color energy $E_c(G)$ for a colored graph G as the sum of the absolute values of color eigenvalues of the color matrix of G.

For the last few years the concept of color energy has been considerably studied and new variations of the same have been introduced (Adiga, Sampathkumar and Sriraj 2013, Betageri 2016, Bhat and D'Souza 2015, Bhat and D'Souza 2017, Joshi and Joseph 2017a, Joshi and Joseph 2017b, Kanna, Kumar and Jagadeesh 2015, Shigehalli and Betageri 2015).

It has been observed that $E_c(G)$ varies according to the type of coloring. The present paper we would explore $E_c(G)$ for the two types of coloring: proper coloring and $L(h, k)$-

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coloring and denote the corresponding color energy (matrix) by $E_\chi(G)$ $A_\chi(G)$ and $E_{LC}(G)$ $A_{LC}(G)$ respectively. The study includes determination of color energy of a bistar $B_{q,q}$ of order $n = 2q$ and some other families of graphs as well as their color complements. Further, the relation between $E_c(G)$ and some graph theoretic parameters is obtained.

We will be using the following results and definitions in the present paper:

Lemma 1.1. (Cvetkovic, Doob and Sachs 1980) Let $A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ be a symmetric block matrix of order 2 × 2. Then

$$Spec(A) = Spec(A_0 + A_1) \cup Spec(A_0 - A_1).$$

Lemma 1.2. (Cvetkovic, Doob and Sachs 1980). Let *A, B, C, D* be matrices where *A* is an invertible matrix and $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then

$$det\, S = det\, A \times det\left(D - CA^{-1}B\right).$$

Definition 1.3. (Adiga, Sampathkumar, Sriraj and Shrikanth 2013) The color complement $\overline{(G)}_c$ of a colored graph *G* is a graph satisfying the following conditions:

1. u_i and u_j are neighbors in $\overline{(G)}_c$ with $i \neq j$, if they are not neighbors and having different colors in *G*.
2. u_i and u_j are not neighbors in $\overline{(G)}_c$, if they are neighbors with $i \neq j$ in *G* or not neighbors having same colors in *G*.

Definition 1.4. (Chartrand and Zhang 2008) Let *h, k* be two non-negative integers and let *G* be a graph. Then *L (h, k)*-coloring of *G* is a function $c : V(G) \to N \cup \{0\}$ satisfying the following conditions:

1. if v_i and v_j are neighbors in *G*, then $\left|c(v_i) - c(v_j)\right| \geq h$,
2. if v_i and v_j are at a distance 2 in *G*, then $\left|c(v_i) - c(v_j)\right| \geq k$.

In the present paper, by a distar $B_{q,q}$ we mean the graph obtained by attaching central vertices of two copies of star $K_{1,q-1}$ by an edge. Note that $B_{q,q}$ is of order $n = 2$q and size $m = 2q-1$. Figure 1 depicts a bistar $B_{4,4}$.

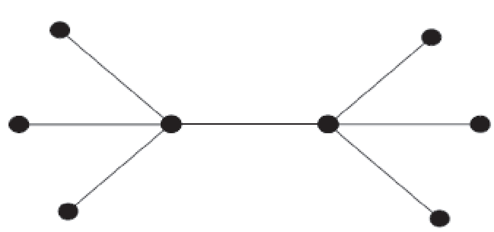

Figure 1: A bistar on 8 vertices, $B_{4,4}$

2. Color Energy of Few Graph Classes and Corresponding Color Complements

Here, we study the color energy of few classes of graphs with proper coloring such as a bistar $B_{q,q}$, $B_{q,q}-$e, the color complement of $B_{q,q}$.

We observe that, $E_\chi\left(B_{q,q}\right) \geq 2(q-1)$. The following theorem gives the exact value of $E_\chi\left(B_{q,q}\right)$.

Theorem 2.1. Let $B_{q,q}$ be a bistar of order greater than 1. Then

$$E_\chi\left(B_{q,q}\right) = 3q - 5 + \left|\frac{-(q-1) + \sqrt{q^2 + 10q - 7}}{2}\right| + \left|\frac{-(q-1) - \sqrt{q^2 + 10q - 7}}{2}\right|.$$

Proof. We observe that $A_\chi\left(B_{q,q}\right)$ is of the form $\begin{pmatrix} D_0 & D_1 \\ D_1 & D_0 \end{pmatrix}$.

Since the characteristic polynomial of $D_0 + D_1$ is

$$P_\chi\left[(D_0 + D_1), \lambda\right] = \left|\lambda I - (D_0 + D_1)\right|,$$

$P_\chi\left[(D_0 + D_1), \lambda\right] = (\lambda - 1) P_\chi(H, \lambda)$, where *N* is a null graph of order $q - 1$.

Thus,

$$P_\chi\left[(D_0 + D_1), \lambda\right] = (\lambda - 1)^{(q-1)}\left[\lambda + (q-2)\right] \quad (1)$$

and $P_\chi\left[(D_0 - D_1), \lambda\right] = \left|\lambda I - (D_0 - D_1)\right|$.

Note that, R_q and C_q denote the q^{th} row and q^{th} column respectively of this determinant. Now we perform some row and column transformations on $\left|\lambda I - (D_0 - D_1)\right|$ as follows.

(i) Add second row to the first row.
(ii) Subtract second row from q^{th} row, for q ≥ 3 and take the factor $\lambda - 1$ from each q^{th} row.
(iii) Add $\sum_{q \geq 3} C_q$ to second column.

Thus,

$$P_\chi(D_0 - D_1, \lambda) = (\lambda - 1)^{(q-2)}\left[\lambda - \frac{\left[-(q-1) + \sqrt{q^2 + 10q - 7}\right]}{2}\right]\left[\lambda - \frac{\left[-(-q-1) - \sqrt{q^2 + 10q - 7}\right]}{2}\right] \quad (2)$$

Hence from Lemma 1.1, equations (1) and (2),

$$Spec_{\chi}\left(B_{q,q}\right) = \begin{pmatrix} -(q-2) & 1 & \frac{-(q-1)+\sqrt{q^2+10q-7}}{2} & \frac{-(q-1)-\sqrt{q^2+10q-7}}{2} \\ 1 & 2q-3 & 1 & 1 \end{pmatrix}$$

and

$$E_{\chi}\left(B_{q,q}\right) = 3q-5+\left|\frac{-(q-1)\pm\sqrt{q^2+10q-7}}{2}\right| + \left|\frac{-(q-1)-\sqrt{q^2+10q-7}}{2}\right|.$$

In (Joshi and Joseph 2017a), we observed that addition of an edge in a graph changes its color energy. Now in the next theorem we examine the effect on color energy with respect to removal of an edge. We consider the graph $B_{q,q}-e$ obtained from the colored graph $B_{q,q}$ by removing the edge joining maximum degree vertices in $B_{q,q}$. It can be verified that $E_{\chi}\left(B_{q,q}-e\right)\geq 2n-7$ and in next theorem, we compute the exact value of $E_{\chi}\left(B_{q,q}-e\right)$.

Theorem 2.2. If $B_{q,q}$ is a bistar of order $n > 1$, e is an edge joining the vertices of maximum degree. Then

$$E_{\chi}\left(B_{q,q}-e\right) = 3q-6+\left|\frac{-(q-1)\pm\sqrt{(q-1)(q+15)}}{2}\right| + \left|\frac{-(q-1)-\sqrt{(q-1)(q+15}}{2}\right|.$$

Proof. The color matrix of $A_{\chi}\left(B_{q,q}-e\right)$ of $B_{q,q}-e$ is of the form $\begin{pmatrix} D_0' & D_1' \\ D_1' & D_0' \end{pmatrix}$ and $D_0'+D_1' = A_{\chi}\overline{\left(K_{1,q-1}\right)_c}$. Therefore, the characteristic polynomial of $D_0'+D_1'$ is

$$P_{\chi}\left[\left(D_0'+D_1'\right),\lambda\right] = \lambda(\lambda-1)^{q-2}\left[\lambda+(q-2)\right] \qquad (3)$$

and the characteristic polynomial of $D_0'-D_1'$ is $P_{\chi}\left[\left(D_0'-D_1'\right),\lambda\right] = \left|\lambda I-\left(D_0-D_1\right)\right|$.

We perform some row and column transformations on $\left|\lambda I-\left(D_0'-D_1'\right)\right|$ as follows.

(i) Subtract second row from q^{th} row, for $q \geq 3$ and take the factor $(\lambda - 1)$ from each q^{th} row.

(ii) Subtract $\sum_{q\geq 3} C_q$ from second column.

Therefore, from Lemma 1.2

$$P_{\chi}\left[\left(D_0'-D_1'\right),\lambda\right] = (\lambda-1)^{(q-2)}\left[\lambda-\frac{-(q-1)+\sqrt{(q-1)(q+15)}}{2}\right]\left[\lambda-\frac{-(q-1)-\sqrt{(q-1)(q+15)}}{2}\right] \qquad (4)$$

Hence from Lemma 1.1, equations (3) and (4),

$$\mathrm{Spec}_{\chi}\left(B_{q,q}-e\right) = \begin{pmatrix} -(q-2) & 0 & 1 & \frac{-(q-1)+\sqrt{(q-1)(q+15)}}{2} & \frac{-(q-1)-\sqrt{(q-1)(q+15)}}{2} \\ 1 & 1 & 2q-4 & 1 & 1 \end{pmatrix}$$

and

$$E_{\chi}\left(B_{q,q}-e\right) = 3q-6+\left|\frac{-(q-1)+\sqrt{(q-1)(q+15)}}{2}\right| + \left|\frac{-(q-1)-\sqrt{(q-1)(q+15)}}{2}\right|$$

Theorem 2.3. Let $B_{q,q}$ be a bistar and $\overline{\left(B_{q,q}\right)_c}$ be its color complement. Then $E_{\chi}\overline{\left(B_{q,q}\right)_c} = 2(2n-5)$.

Proof. The color matrix of $\overline{\left(B_{q,q}\right)_c}$ is $A_{\chi}\left(\overline{\left(B_{q,q}\right)_c}\right) = \begin{pmatrix} O_1 & O_2 \\ O_3 & K \end{pmatrix}$ where O_1, O_2, O_3 are zero matrices of order 2×2, $2 \times (n-2)$, $(n-2)\times 2$ respectively and K is a color matrix of order $(n-2) \times (n-2)$ of a complete bipartite graph $K_{n-2,n-2}$.

Therefore,

$$P_{\chi}\left[\overline{\left(B_{q,q}\right)_c},\lambda\right] = \lambda^2\left[\lambda+2n-5\right](\lambda-1)^{(2n-5)}.$$

Thus,

$$\mathrm{Spec}_{\chi}\overline{\left(B_{q,q}\right)_c} = \begin{pmatrix} 0 & -(2n-5) & 1 \\ 2 & 1 & (2n-5) \end{pmatrix} \text{ and}$$

$$E_{\chi}\overline{\left(B_{q,q}\right)_c} = 2(2n-5).$$

In the next theorem, we consider the graph $\mathcal{S}$ obtained by adding an edge between two pendant vertices of a star $K_{1,n-1}$.

Theorem 2.4. If $\mathcal{S}$ is a colored graph and $\overline{(\mathcal{S})_c}$ is its color complement. Then

$$E_\chi\overline{(\mathcal{S})_c} = n-4+\left|\frac{(n-4)+\sqrt{n^2+56}}{2}\right| + \left|\frac{(n-4)-\sqrt{n^2+56}}{2}\right|.$$

Proof. The color matrix of $\overline{(\mathcal{S})_c}$ is $A_\chi\overline{(\mathcal{S})_c}$ and its characteristic polynomial is $P_\chi\left(\overline{(\mathcal{S})_c},\lambda\right)=\left|\lambda I-\overline{(\mathcal{S})_c}\right|$. After applying a series of row and column transformations, the characteristic polynomial of $A_\chi\overline{(\mathcal{S})_c}$ is

$$P_\chi\left(\overline{(\mathcal{S})_c},\lambda\right) = \lambda^2(\lambda-1)^{(n-4)}\left\{\lambda-\left[\frac{(n-4)+\sqrt{(n-4)^2+4(2n+10)}}{2}\right]\right\} \left\{\lambda-\left[\frac{(n-4)-\sqrt{n^2+56}}{2}\right]\right\}.$$

Hence,

$$\text{Spec}_\chi\overline{(\mathcal{S})_c} = \begin{pmatrix} 0 & 1 & \frac{(n-4)+\sqrt{n^2+56}}{2} & \frac{(n-4)-\sqrt{n^2+56}}{2} \\ 2 & (n-4) & 1 & 1 \end{pmatrix}$$

and

$$E_\chi\overline{(\mathcal{S})_c} = n-4+\left|\frac{(n-4)\pm\sqrt{n^2+56}}{2}\right| + \left|\frac{(n-4)-\sqrt{n^2+56}}{2}\right|.$$

3. Color Energy Associated with L (h, k)-coloring

In this section we study color energy with respect to $L(h, k)$-coloring where h and k as positive intergers. It has been extensively studied due to its wide applications in channel assignment problems (Chartrand and Zhang 2008).

We determine $E_{LC}(G)$ where G is a bistar $B_{q,q}$ and color complement of few families of graphs with diameter 2 and 3. First, we present the result on color energy of a bistar under $L(h, k)$-coloring.

Theorem 3.1. Let $B_{q,q}$ be a bistar with $L(h, k)$-coloring. Then

$$E_{LC}\left(B_{q,q}\right) = n+\sqrt{8n}-4.$$

Proof. Partition the vertex set $B_{q,q}$ into two sets $Y=\{y_1,y_2,\ldots,y_q\}$ and $Z=\{z_1,z_2,\ldots,z_q\}$. Let y_1 and z_1 be the two non-pendant vertices of the $B_{q,q}$. Let $c(y_1)=0$, $c(z_1)=2h$ and $c(y_i)=c(z_i)=h+(i-1)k$, for $2\le i\le q$.

The color matrix of $B_{q,q}$ is $A_{LC}(B_{q,q})$. It is of the form $\begin{pmatrix} A_0' & A_1' \\ A_1' & A_0' \end{pmatrix}$ where A_0' and A_1' are block matrices. Therefore, from Lemma 1.1,

$$Spec_{LC}\left(B_{q,q}\right) = Spec_{LC}\left(A_0'+A_1'\right)\cup Spec_{LC}\left(A_0'-A_1'\right). \quad (5)$$

The characteristic polynomial of $A_0'+A_1'$ is $P_{LC}\left[\left(A_0'+A_1'\right),\lambda\right]=\left|\lambda I-\left(A_0'+A_1'\right)\right|$.

Now apply the following series of row and column transformations on $P_{LC}\left(A_0'+A_1',\lambda\right)$.

(i) Subtract R_2 from R_q, for $q\ge 3$ and take the factor $(\lambda+1)$ common from each q^{th} row,

(ii) Add $\Sigma_{q\ge3}C_q$ to C_2.

Therefore, from the Lemma 1.2

$$P_{LC}\left[\left(A_0'+A_1'\right),\lambda\right]=(\lambda+1)^{q-2}\left(\lambda\pm\sqrt{q}\right). \quad (6)$$

Similarly,

$$P_{LC}\left[\left(A_0'-A_1'\right),\lambda\right]=(\lambda-1)^{q-2}\left(\lambda\pm\sqrt{q}\right). \quad (7)$$

From equations (5), (6) and (7),

$$Spec_{LC}\left(B_{q,q}\right)=\begin{pmatrix} -\sqrt{q} & -1 & 1 & \sqrt{q} \\ 2 & (q-2) & (q-2) & 2 \end{pmatrix}$$

and

$$E_{LC}\left(B_{q,q}\right) = n+\sqrt{8n}-4.$$

Theorem 3.2. Let $B_{q,q}$ be a bistar with $L(h,k)$-coloring with $k\ge h$ and $\overline{\left(B_{q,q}\right)_c}$ be its color complement. Then

$$E_{LC}\overline{\left(B_{q,q}\right)_c} = 2(n-4)$$

Proof. The color matrix of $\overline{\left(B_{q,q}\right)_c}$ is $A_{LC}\left(\overline{\left(B_{q,q}\right)_c}\right)$. If it is expressed in the form $\begin{pmatrix} B_0 & B_1 \\ B_1 & B_0 \end{pmatrix}$, then

$$P_{LC}\left[(B_0+B_1),\lambda\right]=\left|\lambda I-(B_0+B_1)\right|.$$

We apply the row transformations which are listed below:

(i) Add $\sum_{i=2}^{q-1} R_i$ to the first row and take the factor (λ + 2(q−2))common from the first row.

(ii) Subtract 2 × R_1 from i^{th} row, for $i \geq 2$ and take the factor (λ − 2) common from each i^{th} row.

(iii) Subtract $\sum_{i=2}^{q-1} R_i$ from the first row.

Therefore,

$$P_{LC}\left[(B_0+B_1),\lambda\right]=\lambda(\lambda-2)^{(q-2)}\left[\lambda+2(q-2)\right]. \quad (8)$$

The characteristic polynomial for $B_0 - B_1$ is ,

$$P_{LC}\left[(B_0-B_1),\lambda\right]=\lambda^q \quad (9)$$

From Lemma 1.1, equations (8) and (9),

$$Spec_{LC}\overline{\left(B_{q,q}\right)_c}=\begin{pmatrix}0 & 2 & -2(q-2)\\ (q+1) & (q-2) & 1\end{pmatrix}$$

and

$$E_{LC}\overline{\left(B_{q,q}\right)_c}=2(n-4).$$

In the next theorem, we consider the crown graph $S_n^{\,0}$ of order $2n$ as defined in (Adiga, Sampathkumar, Sriraj and Shrikanth 2013). We have derived $E_{LC}\left(S_n^0\right)$ in (Joshi and Joseph 2017b). Now we derive the color energy of its color complement with respect to $L(h, k)$-coloring.

Theorem 3.3. If $S_n^{\,0}$ is a crown graph of order $2n$ with $L(h, k)$-coloring and $\overline{\left(S_n^0\right)_c}$ is its color complement, then $E_{LC}\overline{\left(S_n^0\right)_c}=4(n-1)\cdot$

Proof. The color matrix of $\overline{\left(S_n^0\right)_c}$ is $A_{LC}\left(\overline{\left(S_n^0\right)_c}\right)$ and it can be represented as $\begin{pmatrix}L_0 & L_1\\ L_1 & L_0\end{pmatrix}=\begin{pmatrix}\mathrm{J}-\mathrm{I} & -\mathrm{I}\\ -\mathrm{I} & \mathrm{J}\end{pmatrix}$, where the matrix J is of order $n \times n$ whose all entries are one's.

Here, $L_0+L_1=J-2I$. Therefore, $P_{LC}\left[(J-2I),\lambda\right]=\left|\lambda I-(J-2I)\right|\cdot$

We perform the following row transformations on $P_{LC}\left[(J-2I),\lambda\right]$:

(i) Add $\sum_{i=2}^{n} R_i$ to the first row and from R_1 take [λ − (n − 2)] common.

(ii) $R_1+\sum_{i\geq 2} R_i$ and take (λ + 2) common from each i^{th} row.

(iii) Subtract $\sum_{i=2}^{n} R_i$ from the first row.

Thus,

$$P_{LC}\left[(J-2I),\lambda\right]=\left[\lambda-(n-2)\right](\lambda+2)^{(n-1)}. \quad (10)$$

Now we apply the same series of row transformations on the characteristic polynomial of L_0-L_1, that is $P_{LC}(J,\lambda)$. Therefore,

$$P_{LC}(J,\lambda)=|\lambda I-J|=(\lambda-n)\lambda^{(n-1)}. \quad (11)$$

From Lemma 1.1, equations (10) and (11)

$$Spec_{LC}\overline{\left(S_n^0\right)_c}=\begin{pmatrix}-2 & 0 & n-2 & n\\ (n-1) & (n-1) & 1 & 1\end{pmatrix}$$

and

$$E_{LC}\overline{\left(S_n^0\right)_c}=4(n-1).$$

It has been shown in (Joshi and Joseph 2017b) that the color energy of $K_{1,\,n-1}$ with $L(h,k)$-coloring $E_{LC}(K_{1,\,n-1})$, is same as its energy, $E(K_{1,\,n-1})$. Now we determine color energy of its color complement.

Theorem 3.4. Let $K_{1,\,n-1}$ be a star under $L(h,k)$-coloring with $k \geq h$ and $\overline{\left(K_{1,n-1}\right)_c}$ be its color complement. Then $E_{LC}\overline{\left(K_{1,n-1}\right)_c}=2(n-2)\cdot$

Proof. The color matrix of $\overline{\left(K_{1,n-1}\right)_c}$ is $A_{LC}\overline{\left(K_{1,n-1}\right)_c}$, then its characteristic polynomial is

$$P_{LC}\left(\overline{\left(K_{1,n-1}\right)_c},\lambda\right)=\left|\lambda I-\overline{\left(K_{1,n-1}\right)_c}\right|=\lambda(\lambda+1)^{n-2}\left[\lambda-(n-2)\right].$$

Hence,

$$Spec_{LC}\overline{\left(K_{1,n-1}\right)_c}=\begin{pmatrix}-1 & 0 & (n-2)\\ (n-2) & 1 & 1\end{pmatrix}$$

and

$$E_{LC}\overline{\left(K_{1,n-1}\right)_c}=2(n-2).$$

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Remark 3.5.

(1) $E_{LC}\overline{\left(K_{1,n-1}\right)}_c < E_\chi\left(K_{1,n-1}\right) = E_\chi\left(K_n\right)$.

(2) For the color complement $\overline{\left(K_n\right)}_c$ of a complete graph $K_n, E_{LC}\overline{\left(K_n\right)}_c = E_\chi\overline{\left(K_n\right)}_c$ and for the color complement $\overline{(N)}_c$ of a null graph $N, E_{LC}\overline{(N)}_c = E_\chi(N)$.

4. Color Energy and Graph Theoretic Parameters

Now, we attempt to establish bounds for $E_c(G)$ in terms of few well-known parameters, such as the chromatic number $\chi(G)$, the independence number $\alpha(G)$, the domination number $\gamma(G)$ and maximum degree $\Delta(G)$.

Note that, $m_c^{'}$ denote the number of pairs of non-adjacent vertices of the given colored graph *G*.

We begin with some propositions. Proposition 4.1can be obtained directly from Theorem 2.25 in (Hedetniemi,Slater and Haynes 2013) and Corollary 10 in (Joshi and Joseph 2017a), and Proposition 4.2 can be obtained directly from Theorem 2.1 in (Hedetniemi,Slater and Haynes 2013) and Theorem 6 in (Joshi and Joseph 2017b).

Proposition 4.1. If G is a graph with $\gamma\left(\overline{G}\right) \geq 3$ then $E_{LC}(G) = E(G)$.
Proposition 4.2. If G is a simple graph of order n, then

$$E_c(G) \geq \frac{\gamma(G)\sqrt{2\left(m + m_c^{'}\right)}}{n}$$

The following results follow directly from the Theorems 2.4 and 2.5 in (Adiga, Sampathkumar, Sriraj and Shrikanth 2013), Theorem 2.11 in (Hedetniemi,Slater and Haynes 2013), Theorem 6.10 in (Chartrand and Zhang 2006) and the fact that chromatic number is always less than or equal to number of vertices of corresponding graph, after careful computations.

Theorem 4.3. Let G be a colored graph, then

1. $E_c(G) \leq \left[2_\chi(G)\alpha(G)\left(m + m_c^{'}\right)\right]^{\frac{1}{2}}$.

2. $E_c(G) \leq \left[n\left(n^2 - n + 2m_c^{'}\right)\right]^{\frac{1}{2}}$

3. $E_c(G) \leq \left[2(1 + \Delta(G)\gamma(G)\left(m + m_c{'}\right)\right]^{\frac{1}{2}}$

Theorem 4.4. Let G be a colored graph, then

1. $E_c(G) \geq \left\{\chi(G)\left[\chi - 1\right]D^{\left(\frac{2}{\chi(G}\right)} + 2(m + m_c^{'})\right\}^{\frac{1}{2}}$

2. $E_c(G) \geq \left\{[\Delta(G) + \gamma(G)][\Delta(G) + \gamma(G) - 1]D^{\left(\frac{2}{\Delta(G)+\gamma(G)}\right)} + 2(m + m_c^{'})\right\}^{\frac{1}{2}}$

3. $E_c(G) \geq \left\{2(m + m_c^{'}) + 2\gamma(G)[2\gamma(G) - 1]D^{\left(\frac{1}{\gamma(G)}\right)}\right\}^{\frac{1}{2}}$ provided G has no isolated vertices.

Conclusion

Our exploration of color energy of few bipartite graphs and their color complement has thrown light into several new problems for further study. As observed the color energy with respect to *L(h, k)*-coloring of graph having diameter at most two is the same as its energy. Hence, it is natural to ask whether this case occurs for any graphs having diameter more than two.

Another interesting observation is that there are classes of graphs with the property that $E_{LC}\overline{(G)}_c = E_\chi(G)$. Characterization of graphs satisfying this relation is a problem worth trying. In the same direction, we can also study graphs such that $E_{LC}\overline{(G)}_c = E_\chi\overline{(G)}_c$ and examine whether the same holds good for all types of proper coloring.

A much more general problem that is not yet settled is the characterization of graphs whose color energy does not exceed $2(n-1)$.

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