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A New Proof of the Lester's Perimeter Theorem in Euclidean Space

O. Demirel*, **L. Aslan and D. Topal**

Afyon Kocatepe University, Department of Mathematics, Ahmet Necdet SEZER Campus, 03200, Afyonkarahisar, Turkey

Email: odemirel@aku.edu.tr*, leylaaslan3356@gmail.com, damla.1tpl@gmail.com

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ABSTRACT

An injection defined from Euclidean n-space $E^n(2 \le n < \infty)$ to itself which preserves the triangles of perimeter 1 is an Euclidean motion. J. Lester presented two different proofs for this theorem in Euclidean plane (Lester 1985) and Euclidean space (Lester 1986). In this study we present a general proof which works both in Euclidean plane ($n = 2$) and Euclidean space ($2 < n < \infty$).

1. Introduction

It is well known that some geometric transformations can be characterized by the properties of they preserve. For instance, collinearity preserving bijections of Euclidean n-space E^n $(2 \le n < \infty)$ characterize the affine transformations and this theorem is known as the fundamental theorem of affine geometry. The Möbius transformations of the extended complex plane can be characterized by as transformations preserving quadruples of concylic points. In Minkowski space the Alexandrov's theorem which describes Lorentz transformations as the transformations of Minkowski space preserving the speed of light. In Euclidean space E^n $(2 \le n < \infty)$ the Beckman-Quarles theorem which identifies as motions those functions from E^n to itself preserving pairs of points of a given fixed distance apart. More precisely Beckman-Quarles theorem (Beckman and Quarles 1953) states that a function from E^n to itself which preserves the relation $|x - y| = Q$ for a fixed $Q \in \mathbb{R}^+$ must be an Euclidean motion where $|x - y|$ denotes the distance between $x, y \in E^n$. This theorem plays a major role in our result. G. Martin (unpublished) characterized the equiaffine transformations (affine and area preserving) of E^2 via the injections which preserves triangles with area 1 as follows, see (Lester 1985).

Theorem 1.1: An injection from Euclidean plane to itself which preserves triangles with area 1 must be equiaffine, see (Lester 1985).

J. Lester generalized this theorem to the Euclidean space E^n as follows:

Theorem 1.2: Let f be an injection from Euclidean space E^n $(2 \le n < \infty)$ to itself which preserves triangles with area 1 must be a Euclidean motion, see (Lester 1986).

J. Lester also obtained the following results using triangles of perimeter 1 instead of triangles of area 1.

Theorem 1.3: Let f be an injection from Euclidean plane E^2 to itself which preserves triangles of perimeter 1 must be a Euclidean motion, see (Lester 1985).

Theorem 1.4: Let f be an injection from Euclidean space $E^n(2 \le n < \infty)$ to itself which preserves triangles of perimeter 1 must be a Euclidean motion, see (Lester 1986).

2. A New Proof of the Lester's Perimeter Theorem in Euclidean Space

Lemma 2.1: Let $F_1 = (c, 0, \cdots, 0)$ and $F_2 = (-c, 0, \cdots, 0)$ be two distinct points in Euclidean space E^n, where $0 < c < \frac{1}{4}$ and $2 < n < \infty$. If XF_1F_2 is a triangle of perimeter 1 then must be a point on n-dimensional rotated ellipsoid with equation

$$\frac{x_1^2}{a^2} + \sum_{i=2}^{n} \frac{x_i^2}{b^2} = 1$$

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where $a+c=\frac{1}{2}$, $a^2=b^2+c^2$ and x_i is the natural coordinate function defined by $x_i(p_1,p_2,\cdots,p_n)=p_i$.

Proof: Let us take a point $X=(x_1,x_2,\cdots,x_n)$ such that

$$|X-F_1|+|X-F_2|+|F_1-F_2|=1.$$

Clearly

$$|X-F_1|+|X-F_2|=1-2c=2a$$

which implies $a+c=\frac{1}{2}$. Hence we get

$$\sqrt{\sum_{i=2}^{n}x_i^2+(x_1-c)^2}+\sqrt{\sum_{i=2}^{n}x_i^2+(x_1+c)^2}=2a$$

which implies

$$a^2\left(a^2-c^2\right)=x_1^2\left(a^2-c^2\right)+a^2\sum_{i=2}^{n}x_i^2.$$

Let's denote the number a^2-c^2 by b^2. Clearly we get

$$a^2b^2=x_1^2b^2+a^2\sum_{i=2}^{n}x_i^2$$

and this yields the equation which we desired.

Clearly, $X\neq(\mp a,0,\cdots,0)$ since XF_1F_2 is a triangle of perimeter 1. Notice that the points $A_1=(a,0,\cdots,0)$, $A_2=(-a,0,\cdots,0)$, $F_1=(c,0,\cdots,0)$ and $F_2=(-c,0,\cdots,0)$ are collinear.

Corollary 2.1. Let F_1 and F_2 be two distinct points in Euclidean space $E^n(2<n<\infty)$ such that

$$|F_1-F_2|=2c<\frac{1}{2}$$

and define the set

$$\Omega=\left\{X\in E^n : XF_1F_2 \text{ is a triangle of perimeter } 1\right\}.$$

Then the locus of all points $X\in E^n$ is an *n*-dimensional rotated ellipsoid drilled by two points. These two points are clearly the vertices of the ellipsoid.

Lemma 2.2. Let *f* be an injection from Euclidean space $E^n(2<n<\infty)$ to itself which preserves triangles of perimeter 1. Then *f* preserves the right angles.

Proof: Let l_1 and l_2 be two distinct lines in E^n which meets perpendicularly. Denote the common point of these lines by F_1. Now take a point on l_2, say F_2, such that

$$0<|F_1-F_2|=2c<\frac{1}{2}.$$

Now following the same way in the proof of *Lemma 2.1,* one can easily construct the *n*-dimensional rotated ellipsoid Ω with focal points F_1 and F_2. Clearly l_1 and Ω meets at two points, say *A* and *B*. Now draw the Euclidean line passing through F_2 and parallel to L_1. Obviously, this line and Ω meets at two points and denote them by *C* and *D*. It is clear that either $AC\parallel BD$ or $AD\parallel BC$. Without loss of generality we may assume $AC\parallel BD$. Clearly one can easily see that AF_1F_2C is a rectangle which consists of four triangles AF_1F_2, F_1F_2C, F_2CA and CAF_1. The perimeter of these triangles is 1. Clearly, by hypothesis, the perimeter of the triangles $A'F_1'F_2', F_1'F_2'C', F_2'C'A'$ and $C'A'F_1'$ is 1, where $f(A)=A', f(C)=C', f(F_1)=F_1', f(F_2)=F_2'$. This implies that $A'F_1'F_2'C'$ is a rectangle, see (Lester 1985). Finally one can easily see that $A'F_1'\perp F_1'F_2'$. This yields us $f(l_1)\perp f(l_2)$ which finishes the proof.

Lemma 2.3: Let *f* be an injection from Euclidean space $E^n(2<n<\infty)$ to itself which preserves triangles of perimeter 1 and *a, c* be two positive real numbers satisfying $a+c=\frac{1}{2}$ and $0<c<\frac{1}{4}<a<\frac{1}{2}$. If *A* and *B* are two points satisfying $|A-B|=1-\frac{c}{a}$ then *f* preserves the midpoint of *A* and *B*.

Proof: Let F_1 and F_2 be two distinct points in E^n satisfying $0<|F_1-F_2|<\frac{1}{2}$. Now construct the *n*-dimensional rotated ellipsoid Ω with focal points F_1, F_2 in the same way as in the proof of *Lemma 2.1* and *Corollary 2.1*. Let *K* and *L* be the vertices of Ω. Then for each point *X* of Ω such that $K\neq X\neq L$, the perimeter of the triangle XF_1F_2 is 1. Clearly, there exists appropriate points of Ω to get an orthogonal basis of E^n. More precisely, there exists some points on Ω, say $X_1,X_2,\cdots,X_{n-1}$ such that the set

$$\theta_1=\left\{\overrightarrow{F_1F_2},\overrightarrow{F_1X_1},\overrightarrow{F_1X_2},\cdots,\overrightarrow{F_1X_{n-1}}\right\}$$

is an orthogonal basis of E^n. Clearly $K\neq X_j\neq L$ for each $j\in\{1,2,\cdots,n-1\}$. In addition to this the set

$$\theta_2=\left\{\overrightarrow{F_1F_2},\overrightarrow{F_1Y_1},\overrightarrow{F_1X_2},\cdots,\overrightarrow{F_1X_{n-1}}\right\}$$

is also an orthogonal basis of E^n where $\overrightarrow{F_1Y_1}=-\overrightarrow{F_1X_1}$. Since *f* preserves the right angles by *Lemma 2.2*, it is clear that the sets

$$\theta_1'=\left\{\overrightarrow{F_1'F_2'},\overrightarrow{F_1'X_1'},\overrightarrow{F_1'X_2'},\cdots,\overrightarrow{F_1'X_{n-1}'}\right\}$$

and

$$\theta_2'=\left\{\overrightarrow{F_1'F_2'},\overrightarrow{F_1'Y_1'},\overrightarrow{F_1'X_2'},\cdots,\overrightarrow{F_1'X_{n-1}'}\right\}$$

are also orthogonal bases of E^n where $f(Y_1)=Y_1'$, $f(F_i)=F_i'$ and $f(X_j)=X_j'$ for $\forall i\in\{1,2\}$ and for $\forall j\in\{1,2,\cdots,n-1\}$. Since f is injective and it preserves the triangles of perimeter 1, one can easily see that $\overrightarrow{F_1'Y_1'}=-\overrightarrow{F_1'X_1'}$ and this implies that F_1' is the midpoint of X_1' and Y_1'.

Corollary 2.2. Let f be an injection from Euclidean space $E^n\,(2<n<\infty)$ to itself which preserves triangles of perimeter 1. Then f preserves the Euclidean lines, i.e. f is affine.

Now we are ready to give the general proof of *Theorem 1.4* for $2\le n<\infty$.

Proof of Theorem 1.4: Let ABC be an equailateral triangle with perimeter 1 and denote the midpoints of the sides of ABC by M_{AB}, M_{BC} and M_{AC}. Clearly $AM_{BC}\perp BC$, $BM_{AC}\perp AC$ and $CM_{AB}\perp AB$. Since f preserves the right angles by *Lemma 2.2*, we get $A'M'_{BC}\perp B'C'$, $B'M'_{AC}\perp A'C'$ and $C'M'_{AB}\perp A'B'$ where $M'_{AB}=f(M_{AB}), M'_{AC}=f(M_{AC})$ and $M'_{BC}=f(M_{BC})$. Moreover by *Lemma 2.3*, M'_{AB}, M'_{BC}, and M'_{AC} are the midpoints of the sides of $A'B'C'$ More precisely

$$M'_{AB}=\frac{A'+B'}{2}, M'_{BC}=\frac{B'+C'}{2}, M'_{AC}=\frac{A'+C'}{2}$$

holds. This yields us that the triangle $A'B'C'$ must be an equilateral triangle of perimeter 1. Hence we get $|A'-B'|=|A'-C'|=|B'-C'|=\frac{1}{3}$. Finally we see that f preserves the distance $\frac{1}{3}$. From the Beckman-Quarles theorem (Beckman and Quarles 1953) f must be a motion of E^n. It is easy to see that the method we used in the proof of *Theorem 1.4* is valid for the proof of *Theorem 1.3*.

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Chitkara University, Saraswati Kendra, SCO 160-161, Sector 9-C, Chandigarh, 160009, India

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