



Gaussian Neighborhood-prime Labeling of Graphs Containing Hamiltonian Cycle

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ARTICLE INFORMATION

Received: 23 November, 2018

Revised: 13 December, 2018

Accepted: 27 January, 2019

Published online: March 6, 2019

Keywords:

Gaussian integers, spiral order of Gaussian integers, Gaussian neighborhood-prime labeling.

AMS Subject Classification (2010): 05C78

DOI: <https://doi.org/10.15415/mjis.2019.72021>

ABSTRACT

In this paper, we examine Gaussian neighborhood-prime labeling of generalized Petersen graph and graphs which contain Hamiltonian cycle.

1. Introduction

The extension of prime labeling on natural numbers to the set of Gaussian integer is known as Gaussian prime labeling. The concept of Gaussian prime labeling with the help of spiral ordering of Gaussian integer was firstly introduced by Hunter Lehmann and Andrew Park in (Lehmann and Park 2016). In this paper, Hunter Lehmann and Andrew Park gave a milestone result that any tree with ≤ 72 vertices is Gaussian prime tree under the spiral order. Steven Klee, Hunter Lehmann and Andrew Park (Klee, Lehmann and Park 2016) also proved that the path graph, star graph, n -centipede tree, (n, m, k) double star tree, $(n, 3)$ firecracker tree are Gaussian prime graphs. We all know the Entringer (Robertson and Small 2009) conjecture that any tree admits prime labeling, but this conjecture has not yet been proven for all trees. S.K. Patel and N. P. Shrimali (Patel and Shrimali 2015) introduced one of the variation of prime labeling, which is neighborhood-prime labeling of a graph. They proved that following graphs are neighborhood-prime: path, complete, wheels, Helms, closed Helm, flowers, certain union of cycles. (Patel 2017) proved that Generalized Petersen graphs are neighborhood-prime graphs for certain cases. Malori Cloys and N. Bradley Fox (Cloys and Fox 2018) almost covered large class of trees which have neighborhood-prime labeling such as caterpillars, spiders, firecrackers and any tree that contains no two degree vertices.

In addition, Malori Cloys and N. Bradley Fox put forth conjecture that all trees are neighborhood-prime. Similar conjecture was made by Entringer for prime labelings. John Asplund, N. Bradley Fox and Arran Hamm (Asplund and Fox 2018) well-built the result that any graph containing Hamiltonian cycle is neighborhood-prime graph. With the help of Hamiltonicity, they proved that the Generalized Petersen graph $GP(n, k)$ is neighborhood-prime graph for all n and k . The detailed list of neighborhood-prime graph is also available in the dynamic survey of graph labeling written by (Gallian 2016).

The Gaussian neighborhood-prime labeling was firstly introduced by (Rajesh Kumar and Mathew Varkey 2018) with respect to spiral order. Rajesh Kumar *et al.* initiated their work by showing the graphs like: path, star, (p, n, m) double star tree with $n \leq m$, $\text{comb } P_n \odot K_1$, Spiders, $(n, 2)$ centipede tree, cycles C_n with $n \not\equiv 2 \pmod{4}$ are Gaussian neighborhood-prime graph under spiral ordering of Gaussian integers. In this paper, we investigate Gaussian neighborhood-prime labeling of graph containing Hamiltonian cycle. We will discuss more results depending upon the Hamiltonicity which guarantees that the graph is Gaussian neighborhood-prime graph under the spiral order. Further, we will prove that generalized Petersen graphs are Gaussian neighborhood-prime graphs under the spiral order.

We begin with some definitions and the background of Gaussian integer before introducing main results. We will use spiral ordering of Gaussian integers and its properties given by (Steven Klee *et al.* 2016).

2. Background of Gaussian Integer and Spiral Ordering

The complex numbers of the form $\gamma = p + iq$; $p, q \in \mathbb{Z}$ are known as **Gaussian integer**. We denote set of Gaussian

$$\gamma_{n+1} = \begin{cases} \gamma_n + i & \text{if } \Re(\gamma_n) \equiv 1 \pmod{2}, \Re(\gamma_n) > \Im(\gamma_n) + 1 \\ \gamma_n - 1 & \text{if } \Im(\gamma_n) \equiv 0 \pmod{2}, \Re(\gamma_n) \leq \Im(\gamma_n) + 1, \Re(\gamma_n) > 1 \\ \gamma_n + 1 & \text{if } \Im(\gamma_n) \equiv 1 \pmod{2}, \Re(\gamma_n) < \Im(\gamma_n) + 1 \\ \gamma_n + i & \text{if } \Im(\gamma_n) \equiv 0 \pmod{2}, \Re(\gamma_n) = 1 \\ \gamma_n - i & \text{if } \Re(\gamma_n) \equiv 0 \pmod{2}, \Re(\gamma_n) \geq \Im(\gamma_n) + 1, \Im(\gamma_n) > 0 \\ \gamma_n + 1 & \text{if } \Re(\gamma_n) \equiv 0 \pmod{2}, \Im(\gamma_n) = 0 \end{cases} \quad (1)$$

where γ_n denote the n^{th} Gaussian integer with above ordering. In notation, we write first ‘ n ’ Gaussian integers by $[\gamma_n]$.

In (Steven Klee *et al.* 2016) had already established some useful properties about Gaussian integers with the above ordering like :

- Any two consecutive integers are relatively prime.
- Any two consecutive odd integers are relatively prime.
- γ and $\gamma + \mu(1+i)^k$ are relatively prime, if γ is an odd Gaussian integer and μ is a unit. where k is a positive integer.

Definition 2.1: (Gross and Yellen 1999) *The set of all vertices in G which are adjacent to u is called **neighborhood of vertex u** . In notation, we write $N(u)$.*

Definition 2.2 : (Rajesh Kumar and Mathew Varkey 2018) *Let G be a graph having n vertices. A bijective function $g : V(G) \rightarrow [\gamma_n]$ is called **Gaussian neighborhood-prime labeling**, if the Gaussian integers in the set $\{g(u) : u \in N(w)\}$ are relatively prime for every vertices $w \in V(G)$ with degree greater than one. A graph which admits Gaussian neighborhood-prime labeling is known as **Gaussian neighborhood-prime graph**.*

Definition 2.3 : (Cloys and Fox 2018) *The size of largest cycle in graph G is called **circumference** of a graph G .*

In this paper, we considered all graphs which are undirected, finite and simple. For the notations and terminology of graph theory, we have referred (Gross and Yellen 1999). Throughout this paper, we will understand that the graph is a Gaussian neighborhood-prime graph

integers by $\mathbb{Z}[i]$. An integer γ is called even if it is a multiple of $1 + i$. Otherwise, it is an odd. A **norm** on $\mathbb{Z}[i]$ is defined to be $d(p + iq) = p^2 + q^2$. The only units of $\mathbb{Z}[i]$ are $\pm 1, \pm i$. The associates of γ are unit multiple of γ . In $\mathbb{Z}[i]$, γ and γ' are relatively prime if the only units are common divisors of γ and γ' . A γ is said to be prime if and only if the only divisors of γ are $\pm 1, \pm i, \pm \gamma, \pm i\gamma$.

(Steven Klee *et al.* 2016) introduced the **Spiral ordering of the Gaussian integer** and defined γ_{n+1} recursively starting with $\gamma_1 = 1$ as follows:

meant to be a Gaussian neighborhood-prime graph with the spiral ordering.

3. Main Results

Theorem 3.1 *If H is a Hamiltonian graph having n vertices with $n \not\equiv 2 \pmod{4}$, then H is a Gaussian neighborhood-prime graph.*

Proof: Firstly, we note that if H is a Gaussian neighborhood-prime graph then the new graph formed by adding an edge in H between two vertices of degree at least two is again Gaussian neighborhood-prime graph.

In order to prove H has Gaussian neighborhood-prime labeling it is enough to show that the Hamiltonian cycle of H has Gaussian neighborhood-prime labeling. Let $C = (v_1, v_2, \dots, v_n)$ be the Hamiltonian cycle in graph H . Now, we reformulate the labeling of cycles used by (Rajesh Kumar and Mathew Varkey 2018) in as follows:

Define a bijection $f : V(H) \rightarrow [\gamma_n]$ by

$$f(v_i) = \begin{cases} \gamma_{\lfloor \frac{n}{2} \rfloor + \frac{i+1}{2}} & \text{if } i \text{ is odd} \\ \gamma_{\frac{i}{2}} & \text{if } i \text{ is even} \end{cases} \quad (2)$$

Note that each vertex $v_i (i \neq 1)$ of C there exist two neighbors of v_i whose labels are consecutive Gaussian integers. The neighbors of v_1 are v_2, v_n , having labels γ_1, γ_n respectively. Hence, Hamiltonian cycle C has Gaussian neighborhood-prime labeling if n is not congruent to 2 modulo 4. Which completes the proof.

Theorem 3.2 Let \bar{H} be a Gaussian neighborhood-prime graph with n vertices, where n is not congruent to 2 modulo 4, having Hamiltonian cycle $C = (u_1, u_2, \dots, u_n)$. If graph H is obtained from \bar{H} using additional k vertices $\{w_1, w_2, \dots, w_k\}$ in such a way that each w_j is adjacent to u_{m_i} and u_{m_i+2} where subscripts of $V(C)$ are calculated under modulo n then H is Gaussian neighborhood prime graph.

Proof: We define the labeling $h: V(G) \rightarrow [\gamma_n]$ by $h(u_i) = f(u_i)$ where $f(u_i)$ is defined in Equation 2 and $h(w_i)$ arbitrarily from the set $\{\gamma_{n+1}, \gamma_{n+2}, \dots, \gamma_{n+k}\}$. Since $N_{\bar{H}}(u_i) \subseteq N_H(u_i)$, it follows that the labels of $N_H(u_i)$ contains consecutive Gaussian integers So, we only need to check that the integers in the set $\{h(w_i) | w \in N(w_i)\}$ are relatively prime. As each w_j is adjacent to u_{m_i} and u_{m_i+2} which are either labeled by consecutive Gaussian integers or one of them contains the label γ_1 . Thus, h is a Gaussian neighborhood-prime labeling. Consequently, H is a Gaussian neighborhood-prime graph.

Theorem 3.3 If H is a Hamiltonian graph having an odd cycle then H has Gaussian neighborhood-prime labeling.

Proof: Let H be a Hamiltonian graph having n vertices.

Case I: $n \not\equiv 2 \pmod{4}$

By Theorem 3.1, H has Gaussian neighborhood prime labeling.

Case II: $n \equiv 2 \pmod{4}$

Let $C = (u_1, u_2, \dots, u_n)$ be a Hamiltonian cycle of H that contains an odd cycle. There must be chord in H which forms an odd cycle with length m using $m - 1$ consecutive edges from the Hamiltonian cycle of C . Without loss of generality, let $u_1 u_m$ be the chord such that $C' = (u_1, u_2, \dots, u_m)$ forms an odd cycle. We assign the labels to vertices $u_1, u_3, u_5, \dots, u_{n-1}$ by $\gamma_1, \gamma_2, \dots, \gamma_n$ and $u_{m+1}, u_{m+3}, u_{m+5}, \dots, u_n, u_2, u_4, \dots, u_{m-1}$ by $\gamma_{\frac{n}{2}+1}, \gamma_{\frac{n}{2}+2}, \dots, \gamma_n$. Obviously, $u_1 \in N(u_m)$ having label γ_1 . So, we are done. In $N(u_i)$ ($i \neq m$), there are two vertices whose labels are consecutive Gaussian integers. Hence, H admits Gaussian neighborhood-prime labeling.

Theorem 3.4 If G is a connected graph with n vertices such that $n \not\equiv 3 \pmod{4}$ and G has circumference $n - 1$ then G has a Gaussian neighborhood-prime labeling.

Proof: Let $C = (w_1, w_2, \dots, w_{n-1})$ be the cycle and w be the vertex in graph G which does not lie on cycle C . We have following cases for vertex w :

Case-I: If $\deg(w) = 1$ then we define the labeling $h: V(G) \rightarrow [\gamma_n]$ by $h(w) = f(w)$ where $f(w)$ is defined in

Equation 2 (In which n is replaced by $n - 1$) and $h(w) = \gamma_n$. The reader can easily verify that h is a Gaussian neighborhood-prime labeling.

Case-II: If $\deg(w) > 1$ then without loss of generality we assume that w is adjacent to w_k ($1 \leq k \leq n - 1$) on cycle C . Define a bijection $h: V(G) \rightarrow [\gamma_n]$ by

$$h(u_{i+(k-2)}) = f(u_i) = \begin{cases} \gamma_{\lfloor \frac{n-1}{2} \rfloor + \frac{i+1}{2}} & \text{if } i \text{ is odd} \\ \gamma_{\frac{i}{2}} & \text{if } i \text{ is even} \end{cases} \quad (3)$$

and $h(w) = \gamma_n$, where the subscript $i + (k - 2)$ is calculated under modulo $n - 1$.

From Equation (3), one can see that $\gamma_i \in \{h(v) | v \in N(w)\}$. The other vertices of cycle C consists two neighbors whose labels are consecutive Gaussian integers. In both the cases, we have cycle if length $n - 1$ which is Gaussian neighborhood-prime graph if $n - 1 \not\equiv 2 \pmod{4}$. Thus, h admits Gaussian neighborhood-prime labeling if $n \not\equiv 3 \pmod{4}$.

(Cloy and Fox 2018) proved $GP\left(n, \frac{n}{2}\right)$ is a neighborhood-prime graph. We use the idea of the neighborhood-prime labeling of $GP\left(n, \frac{n}{2}\right)$ in the following lemma.

Lemma 3.5 For all $n \geq 8$ with $n \equiv 0 \pmod{4}$, the generalized Petersen graph $GP\left(n, \frac{n}{2}\right)$ has a Gaussian neighborhood-prime labeling.

Proof: Here we define a bijection $h: V(GP\left(n, \frac{n}{2}\right)) \rightarrow [\gamma_{2n}]$ as follows:

$$\begin{aligned} h(u_{1+2t}) &= \gamma_{n+3+4t} & \text{for } 0 \leq t < \frac{n}{2} \\ h(u_{2t}) &= \gamma_{1+4t} & \text{for } 0 \leq t < \frac{n}{2} \\ h(v_{2t}) &= \gamma_{n+2+4t} & \text{for } 0 \leq t < \frac{n}{2} \\ h(v_{1+2t}) &= \gamma_{4+4t} & \text{for } 0 \leq t < \frac{n}{2} \end{aligned}$$

where the subscripts of Gaussian integers are calculated under modulo $2n$.

From the definition of h , note that the interior vertices v_i are only adjacent to u_i and $v_{i+\frac{n}{2}}$ for each $0 \leq i \leq n - 1$. These vertices are labeled by consecutive Gaussian integers.

For each $0 \leq i < n - 1$, the vertex u_i has neighbors v_i and u_{i+1} which are also labeled by consecutive Gaussian integers. The vertices u_{n-1} has neighbor u_0 whose label is γ_1 . Thus, for each $v \in V\left(GP\left(n, \frac{n}{2}\right)\right)$ the set $\{h(u) : u \in N(v)\}$ consists relatively prime Gaussian integers. Therefore, h is a Gaussian neighborhood-prime labeling.

(Patel 2017) proved $GP(n, k)$ is neighborhood-prime graph if n and k are relatively prime, we will use the approach of that labeling in the following lemma.

Lemma 3.6 For each n with $n \equiv 5 \pmod 6$ and $k = \frac{(n-1)}{2}$ or 2, the generalized Petersen graph $GP(n, k)$ has a Gaussian neighborhood-prime labeling.

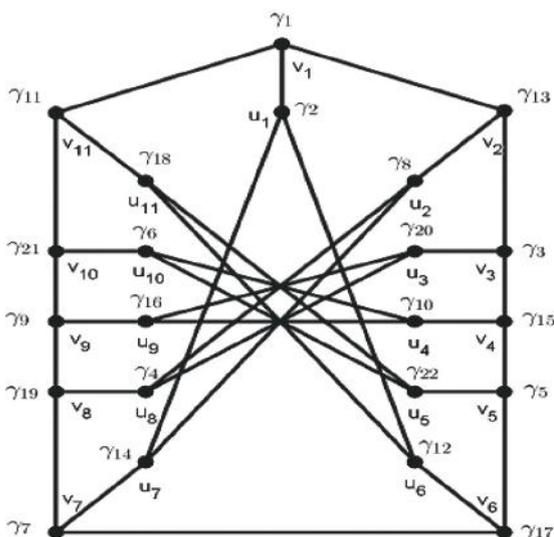
Proof: Let $G = GP(n, k)$. Consider the vertex set $V(G) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$ where the subscripts are reduced modulo n . We define a bijective function $h: V(G) \rightarrow [\gamma_{2n}]$ as follows :

$$\begin{aligned} h(v_{2m+1}) &= \gamma_{2m+1}, \quad 0 \leq m \leq \frac{n-1}{2} \\ h(v_{2m+2}) &= \gamma_{n+(2m+2)}, \quad 0 \leq m \leq \frac{n-3}{2} \\ h(u_1) &= \gamma_2 \end{aligned}$$

We assign the labels of interior vertices u_i (except u_1) in the following manner:

Case-I: $n \equiv 1 \pmod 4$

$$h(u_{2m+1}) = \begin{cases} \gamma_{(2m+1)+k+1} & \text{if } 0 < m \leq \frac{n-1}{4} \\ \gamma_{(2m+1)-k+1} & \text{if } \frac{n-1}{4} < m \leq \frac{n-1}{2} \end{cases} \quad (4)$$



$$h(u_{2m+2}) = \begin{cases} \gamma_{(2m+2)+n+k+1} & \text{if } 0 \leq m \leq \frac{n-5}{4} \\ \gamma_{(2m+2)+n-k+1} & \text{if } \frac{n-5}{4} < m \leq \frac{n-3}{2} \end{cases} \quad (5)$$

Case-II: $n \equiv 1 \pmod 3$

$$h(u_{2m+1}) = \begin{cases} \gamma_{(2m+1)+n+k+1} & \text{if } 0 \leq m \leq \frac{n-3}{4} \\ \gamma_{(2m+1)+n-k+1} & \text{if } \frac{n-3}{4} < m \leq \frac{n-1}{2} \end{cases} \quad (6)$$

$$h(u_{2m+2}) = \begin{cases} \gamma_{(2m+2)+k+1} & \text{if } 0 \leq m \leq \frac{n-3}{4} \\ \gamma_{(2m+2)-k+1} & \text{if } \frac{n-3}{4} < m \leq \frac{n-3}{2} \end{cases} \quad (7)$$

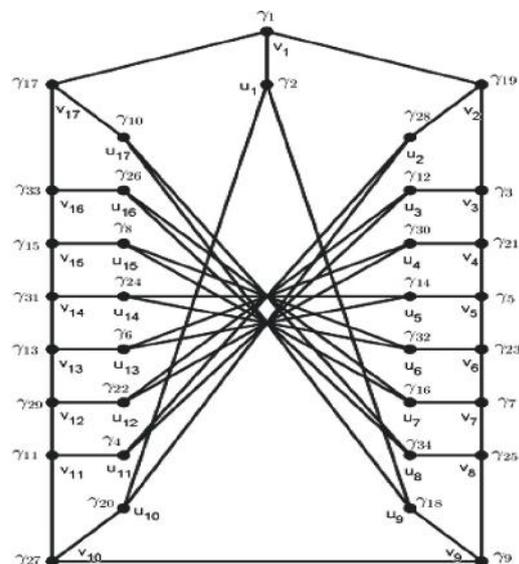
Let w be any vertex of $V(GP(n, k))$, we claim that the Gaussian integers in the set $\{h(x) : x \in N(w)\}$ are relatively prime.

Case-1: $w = v_j, 1 \leq j \leq n$

The neighbors of exterior vertices $v_j (j \neq 2, n)$ are v_{j-1}, v_{j+1} having labels $h(v_{j-1})$ and $h(v_{j+1})$ which are consecutive odd Gaussian integers. The vertices v_2 and v_n have common neighbor v_1 whose label is γ_1 .

Case-2: $w = u_j, 1 \leq j \leq n$

The neighbors of internal vertices u_j are $v_j, u_{j+k}, u_{j+(k+1)}$ where the subscripts of vertices u_j are reduced modulo n . From the Equations (4), (5), (6), (7) observe that for each $u_j (j \neq 1)$, either $h(v_j)$ and $h(u_{j+k})$ is consecutive Gaussian integers or $h(v_j)$ and $h(u_{j+(k+1)})$ are consecutive Gaussian integers. Finally, one of the neighbor of vertex u_1 is v_1 with label γ_1 .



Thus, the set $\{h(x) : x \in N(w)\}$ consists relatively prime Gaussian integers in each of the above cases which implies that h is

a Gaussian neighborhood prime labeling.

Figure 1. Gaussian Neighborhood-prime labeling of the $GP(11,5)$ and $GP(17,8)$

Theorem 3.7 For each n and k , the generalized Peterson graph $GP(n, k)$ has Gaussian neighborhood-prime labeling.

Proof: In (Alspach 1983) proved that for each n and k , the graph $GP(n, k)$ is Hamiltonian except following two cases:

- $n \equiv 0 \pmod{4}$ and $n \geq 8$ with $k = \frac{n}{2}$.
- $n \equiv 5 \pmod{6}$ with $k = \frac{n-1}{2}$ or 2, which are known to be isomorphic graphs.

The proof follows from Theorem 3.1 and Theorem 3.3 together with previous two lemmas.

Acknowledgements

We express our immense gratitude towards the experts, who guided as referee and plotted suggestions, as and when required. Further, we extend our gratitude to DST-FIST.

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