Math. J. Interdiscip. Sci.Vol. 7, No. 2, March 2019, pp.125–134

Mathematical Journal of Interdisciplinary Sciences

Journal homepage: https://mjis.chitkara.edu.in/

Bounds For The Complex Part of Phase Velocity

S Sridevi [1], V Ganesh[2] and R Thenmozhy[3]

[1]*Department of Mathematics, Rajiv Gandhi College of Engineering and Technology, Kirumampakkam, Puducherry, 607 402, India*
[2]*Department of Mathematics, Higher College of Technology, Muscat, Sultanate of Oman*
[3] *Department of Mathematics, Periyar Govt. Arts College, Cuddalore, Tamilnadu, 607 001, India*

[*]Email: jaysridevi@yahoo.co.in

ARTICLE INFORMATION

Received: 20 December, 2018
Revised: 19 January, 2019
Accepted: 14 February, 2019

Published online: March 6, 2019

Keywords:
Hydrodynamics Phase velocity Instability region

DOI: https://doi.org/1015415/mjis.2019.72017

ABSTRACT

We study incompressible, inviscid, density stratified fluid with variable cross section in this present paper. For this problem, we have obtained a bound for the complex part of phase velocity. Furthermore, we have obtained an instability region which depends on number of parameters.

1. Introduction

In physical oceanography, stability of inviscid, incompressible, density stratified fluid with variable topography is an important study. In hydrodynamics, standard Taylor-Goldstein problem has been used to analyse constant topography whereas extended Taylor-Goldstein problem of hydrodynamic stability is employed to study variable topography. The latter study was initiated by (Pratt *et al.* 2000) and the problem have been extended further by Deng. *et al.* (Deng, Pratt, Howard and Jones 2008). (Pratt *et. al.* 2000) stated that it is necessary to know in advance the flow might be stable or unstable and the range of phase speeds. A number of general analytical results have been derived for this problem by the various authors namely, (Deng *et al.* 2008), (Pratt *et al.* 2000), (Reddy and Subbiah 2014) and (Subbiah and Ganesh 2009, Subbiah and Ganesh 2008), (Ganesh 2010), (Sridevi *et. al.* 2017), (Sridevi and Ganesh 2015, Sridevi and Ganesh 2018) and (Reenapriya and Ganesh 2015).

In our present work we have obtained the bounds for the complex part of the phase velocity. Also, we have derived an instability region depending on various parameters.

2. Extended Taylor-Goldstein Equation

The eigen value problemfrom (Deng *et. al.* 2008) is given by

$$\left[\frac{(bW)'}{b}\right]' + \left[\frac{N^2}{(U_0-c)^2} - \frac{b\left(\frac{U_0'}{b}\right)'}{U_0-c} - k^2\right]W = 0, \tag{2.1}$$

with

$$W(0) = 0 = W(D). \tag{2.2}$$

Where, W(z) - eigen function, $c = c_r + ic_i$ complex phase velocity, where $c_i > 0$ for unstable modes $c_i > 0 (< 0)$ for growing (decaying) modes and $N^2(z) > 0$ - stratification parameter, $b(z)$ - breadth function, $U_0(z)$ - basic velocity profile and $k > 0$ - wave number.

3. Bounds for Complex Part of Phase Velocity

3.1 Theorem

If $c = c_r + ic_i$ and imaginary part of $c > 0$ then

$$\int |W'|^2\, dz + \int \mathrm{k}^2 |W| dz - \frac{1}{2}\int T' |W|^2\, dz - \int \frac{N^2}{|U_0-c|^4}\left[(U_0-c_r)^2 - c_i^2\right] |W|^2\, dz + \int \frac{b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^2}(U_0-c_r)\, |W|^2\, dz = 0.$$

ISSN No.: 2278-9561 (Print) ISSN No.: 2278-957X(Online) Registration No. : CHAENG/2013/49583 Periodicity: Bi-Annually

Proof:

Equation (2.1) can also be written as

$$W'' + TW' + T'W + \frac{N^2}{(U_0 - c)^2}W - \frac{b(\frac{U_0'}{b})'}{U_0 - c}W - k^2 W = 0, \tag{3.1}$$

with boundary conditions

$$W(0) = 0 = W(D). \tag{3.2}$$

Multiplying (3.1) by W^*, integrating and applying over boundary conditions (3.2), we get

$$\int\left[|W'|^2 + k^2|W|^2\right]dz - \int TW'W^* dz - \int T'|W|^2\, dz - \int \frac{N^2}{(U_0 - c)^2}|W|^2\, dz + \int \frac{b\left(\frac{U_0'}{b}\right)'}{(U_0 - c)}|W|^2\, dz = 0.$$

Equating real parts, we get

$$\int\left[|W'|^2 + k^2|W|^2\right]dz - \mathrm{Re}.\int TW'W^* dz - \int T'|W|^2\, dz - \int \frac{N^2}{|U_0 - c|^4}\left[(U_0 - c_r)^2 - c_i^2\right]|W|^2\, dz + \int \frac{b\left(\frac{U_0'}{b}\right)'}{|U_0 - c|^2}(U_0 - c_r)|W|^2\, dz = 0.$$

Since Re. $\int TW'W^* dz = -\frac{1}{2}\int T'|W^2|\, dz$, we have [cf. [3]]

$$\int\left[|W'|^2 + k^2|W|^2\right]dz + \frac{1}{2}\int T'|W|^2\, dz - \int T'|W|^2\, dz - \int \frac{N^2}{|U_0 - c|^4}\left[(U_0 - c_r)^2 - c_i^2\right]|W|^2\, dz + \int \frac{b\left(\frac{U_0'}{b}\right)'}{|U_0 - c|^2}(U_0 - c_r)|W|^2\, dz = 0,$$

$$\int\left[|W'|^2 + k^2|W|^2\right]dz - \frac{1}{2}\int T'|W|^2\, dz - \int \frac{N^2}{|U_0 - c|^4}\left[(U_0 - c_r)^2 - c_i^2\right]|W|^2\, dz + \int \frac{b\left(\frac{U_0'}{b}\right)'}{|U_0 - c|^2}(U_0 - c_r)|W|^2\, dz = 0. \tag{3.3}$$

3.2 Theorem

If $c = c_r + ic_i$, then the following integral relation is true

$$\int\left|\left[\frac{(bW)'}{b}\right]'\right|^2 dz - \frac{k^2}{2}\int T'|W|^2\, dz + k^2\int |W'|^2\, dz + k^2\int \frac{N^2}{|U_0 - c|^4}\left[(U_0 - c_r)^2 - c_i^2\right]|W|^2\, dz + 2\int \frac{N^2 b\left(\frac{U_0'}{b}\right)'}{|U_0 - c|^2}(U_0 - c_r)|W|^2\, dz - \int \frac{N^4}{|U_0 - c|^4}|W|^2\, dz - k^2\int \frac{b\left(\frac{U_0'}{b}\right)'}{|U_0 - c|^2}(U_0 - c_r)|W|^2\, dz - \int \frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{|U_0 - c|^2}|W|^2\, dz = 0.$$

Proof

The eigen value problem is given by

$$\left[\frac{(bW)'}{b}\right]' + \left[\frac{N^2}{(U_0 - c)^2} - \frac{b(\frac{U_0'}{b})'}{U_0 - c} - k^2\right]W = 0, \tag{3.4}$$

with boundary conditions

$$W(0) = 0 = W(D). \tag{3.5}$$

Multiplying (3.4) by $\left[\frac{(bW^*)'}{b}\right]'$, integrating and applying over boundary condition (3.5), we get

$$\int\left|\left[\frac{(bW)'}{b}\right]'\right|^2 dz + \int\left[\frac{N^2}{(U_0 - c)^2} - \frac{b\left(\frac{U_0'}{b}\right)'}{U_0 - c} - k^2\right]W\left[\frac{(bW^*)'}{b}\right]' dz = 0. \tag{3.6}$$

From (3.4), we get

$$\left[\frac{(bW^*)'}{b}\right]' = \left[k^2 + \frac{b\left(\frac{U_0'}{b}\right)'}{U_0 - c^*} - \frac{N^2}{(U_0 - c^*)^2}\right]W^*. \tag{3.7}$$

ISSN No.: 2278-9561 (Print) ISSN No.: 2278-957X (Online) Registration No. : CHAENG/2013/49583

Substituting (3.7) in (3.6), we get

$$\int\left|\left[\frac{(bW')'}{b}\right]\right|^2 dz+k^2\int TW^*W'dz+k^2\int|W'|^2\,dz$$
$$+k^2\int\frac{N^2}{(U_0-c)^2}|W|^2\,dz$$
$$+\int\frac{N^2b\left(\frac{U_0'}{b}\right)'}{(U_0-c)^2\left(U_0-c^*\right)}|W|^2\,dz$$
$$-\int\frac{N^4}{|U_0-c|^4}|W|^2\,dz$$
$$-k^2\int\frac{b\left(\frac{U_0'}{b}\right)'}{(U_0-c)}|W|^2\,dz-\int\frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{|U_0-c|^2}|W|^2\,dz$$
$$+\int\frac{N^2b\left(\frac{U_0'}{b}\right)'}{(U_0-c)^2(U_0-c^*)^2}|W|^2\,dz=0;$$

i.e.,

$$\int\left|\left[\frac{(bW')'}{b}\right]\right|^2 dz+k^2\int TW^*W'dz+k^2\int|W'|^2\,dz$$
$$+k^2\int\frac{N^2}{|U_0-c|^4}\left[(U_0-c_r)^2-c_i^2+i2\left(U_0-c_r\right)c_i\right]|W|^2\,dz$$
$$+\int\frac{N^2b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^4}\left[(U_0-c_r)+ic_i\right]|W|^2\,dz-\int\frac{N^4}{|U_0-c|^4}|W|^2\,dz$$
$$-k^2\int\frac{b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^2}\left[(U_0-c_r)+ic_i\right]|W|^2\,dz-\int\frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{|U_0-c|^2}|W|^2\,dz$$
$$+\int\frac{N^2b\left(\frac{U_0'}{b}\right)'}{(U_0-c)^2(U_0-c^*)^2}\left[(U_0-c_r)-ic_i\right]|W|^2\,dz=0.$$

Taking real parts, we get

$$\int\left|\left[\frac{(bW')'}{b}\right]\right|^2 dz-\frac{k^2}{2}\int T|W|^2\,dz+k^2\int|W'|^2\,dz$$
$$+k^2\int\frac{N^2}{|U_0-c|^4}\left[(U_0-c_r)^2-c_i^2\right]|W|^2\,dz$$
$$+2\int\frac{N^2b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^4}\left(U_0-c_r\right)|W|^2\,dz$$
$$-\int\frac{N^4}{|U_0-c|^4}|W|^2\,dz$$
$$-k^2\int\frac{b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^2}(U_0-c_r)|W|^2\,dz$$
$$-\int\frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{|U_0-c|^2}|W|^2\,dz=0. \tag{3.8}$$

3.3 Theorem

If $T'\le 0$, then the bounds for complex part of the phase velocity $c=c_r+ic_i$ is given by

$$c_i\le\left[\frac{N_{\min}^2\left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}\left(\frac{U_{0\max}-U_{0\min}}{2}\right)+N_{\max}^2+\left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}^2\left(\frac{U_{0\max}-U_{0\min}}{2}\right)^2}{\frac{\pi^4b_{\min}^2}{D^4b_{\max}^2}+\frac{2k^2\pi^2}{D^2}+k^4}\right]^{\frac{1}{4}}.$$

Proof:

Multiplying (3.3) by k^2 and adding with (3.8), we get

$$\int\left|\left[\frac{(bW')'}{b}\right]\right|^2 dz+2k^2\int|W'|^2\,dz$$
$$+k^4\int|W|^2\,dz-k^2\int T'|W|^2\,dz$$
$$+2\int\frac{N^2b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^4}\left(U_0-c_r\right)|W|^2\,dz$$
$$-\int\frac{N^4}{|U_0-c|^4}|W|^2\,dz$$
$$-\int\frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{|U_0-c|^2}|W|^2\,dz=0. \tag{3.9}$$

Since $T' \le 0$, dropping the fourth integration and $\frac{(U_0 - c_r)}{|U_0 - c|^2} \le \frac{1}{2c_i}, \frac{1}{|U_0 - c|^2} \le \frac{1}{c_i^2}$ and using Rayleigh - Ritz inequality

$\int |W'|^2 \, dz \ge \frac{\pi^2}{D^2} \int |W|^2 \, dz$ and

$\int \left| \left[\frac{(bW)'}{b} \right]' \right|^2 dz \ge \frac{\pi^4 b_{min}^2}{D^4 b_{max}^2} \int |W|^2 \, dz$ we get

$$\left[\frac{\pi^4 b_{\min}^2}{D^4 b_{\max}^2} + 2k^2 \frac{\pi^2}{D^2} + k^4 \right] \int |W|^2 \, dz$$

$$\le \left| \frac{2N_{\max}^2 \left| b\left(\frac{U_0'}{b} \right)' \right|_{\max}}{2c_i^3} + \frac{N_{\max}^4}{c_i^4} + \frac{\left| b\left(U_0' \right) \right|_{\max}}{c_i^2} \right| \int |W|^2 \, dz,$$

$$c_i^4 \le \frac{\left[N_{max}^2 \left| b\left(\frac{U_0'}{b} \right)' \right|_{max} c_i + N_{max}^4 + \left| b\left(\frac{U_0'}{b} \right)' \right|_{max}^2 c_i^2 \right]}{\left[\frac{\pi^4 b_{min}^2}{D^4 b_{max}^2} + 2k^2 \frac{\pi^2}{D^2} + k^4 \right]}.$$

since $c_i \le \left(\frac{U_{0max} - U_{0min}}{2} \right)$, we get

$$c_i \le \left[\frac{\left[N_{\max}^2 \left| b\left(\frac{U_0'}{b} \right)' \right|_{\max} \left(\frac{U_{0\max} - U_{0\min}}{2} \right) + N_{\max}^4 + \left| b\left(\frac{U_0'}{b} \right)' \right|_{\max}^2 \left(\frac{U_{0\max} - U_{0\min}}{2} \right)^2 \right]}{\left[\frac{\pi^4 b_{\min}^2}{D^4 b_{\max}^2} + 2k^2 \frac{\pi^2}{D^2} + k^4 \right]} \right]^{\frac{1}{4}}.$$

4. Instability Region

4.1 Theorem

If $c_i > 0$ then we have the true intergral relation

$$\int \left[\frac{|(bW)'|^2}{b} + k^2 b |W|^2 \right] dz \le$$

$$[U_{0max} - U_{0min}]^2 \left| \frac{\left| b\left(\frac{U_0'}{b} \right)' \right|_{max} \frac{[U_{0max} - U_{0min}]}{4} + |N^2|_{max} - \frac{|N^2|_{min}}{4}}{c_i^2} \right| \int \frac{b|W|^2}{|U_0 - c|^2} dz.$$

Proof:

Multiplying (2.1) by (bW^*) integrating and applying the boundary conditions(2.2), we get

$$\int \left[\frac{(bW)'}{b} \right]' \left(bW^* \right) dz + \int \frac{N^2}{(U_0 - c)^2} W\left(bW^* \right) dz - \int \frac{b\left(\frac{U_0'}{b} \right)'}{U_0 - c} W\left(bW^* \right) dz - \int k^2 W\left(bW^* \right) dz = 0,$$

$$\int \left[\frac{|(bW)'|^2}{b} + k^2 b |W|^2 \right] dz + \int \frac{b\left(\frac{U_0'}{b} \right)'}{(U_0 - c)} b|W|^2 \, dz - \int \frac{N^2}{(U_0 - c)^2} b|W|^2 \, dz = 0,$$

$$\int \left[\frac{|(bW)'|^2}{b} + k^2 b |W|^2 \right] dz + \int \frac{b\left(\frac{U_0'}{b} \right)'}{(U_0 - c)} \frac{(U_0 - c^*)}{(U_0 - c^*)} b|W|^2 \, dz - \int \frac{N^2}{(U_0 - c)^2} \frac{(U_0 - c^*)^2}{(U_0 - c^*)^2} b|W|^2 \, dz = 0,$$

$$\int \left[\frac{|(bW)'|^2}{b} + k^2 b |W|^2 \right] dz + \int \frac{b\left(\frac{U_0'}{b} \right)' \left(U_0 - (c_r - ic_i) \right)}{|U_0 - c|^2} b|W|^2 \, dz - \int \frac{N^2}{|(U_0 - c|^4} \left(U_0 - (c_r - ic_i) \right)^2 b|W|^2 \, dz = 0.$$

Real parts are separated to get

$$\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz$$
$$+\int\frac{b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{|U_0-c|^2}b|W|^2\,dz$$
$$-\int\frac{N^2\left[(U_0-c_r)^2-c_i^2\right]}{|U_0-c|^4}b|W|^2\,dz=0,$$
$$\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz$$
$$=-\int\frac{b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{|U_0-c|^2}b|W|^2\,dz$$
$$+\int\frac{N^2\left[(U_0-c_r)^2-c_i^2\right]}{|U_0-c|^4}b|W|^2\,dz.$$

Triangular inequality is used to get

$$\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz\le\int\frac{b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{|U_0-c|^2}b|W|^2\,dz$$
$$+\int\frac{N^2\left[(U_0-c_r)^2-c_i^2\right]}{|U_0-c|^4}b|W|^2\,dz.$$

Substituting in the place of $(U_0-c_r)\le[U_{0max}-U_{0min}]$, $\frac{c_i^2}{|U_0-c|^2}\le 1$ and $\frac{1}{|U_0-c|^2}\le\frac{1}{c_i^2}$,

we get,

$$\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz\le$$
$$\left[\left|b\left(\frac{U_0'}{b}\right)'\right|_{max}[U_{0max}-U_{0min}]+\frac{|N^2|_{max}[U_{0max}-U_{0min}]^2}{c_i^2}-\frac{|N^2|_{min}\,c_i^2}{c_i^2}\right]\int\frac{b|W|^2}{|U_0-c|^2}dz,$$
$$\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz\le$$
$$\left[\frac{\left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}[U_{0max}-U_{0min}]c_i^2+|N^2|_{max}[U_{0max}-U_{0min}]^2-|N^2|_{min}\,c_i^2}{c_i^2}\right]\int\frac{b|W|^2}{|U_0-c|^2}dz.$$

By substituting $c_i^2\le\left[\frac{U_{0max}-U_{0min}}{2}\right]^2$ in the above equation

$$\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz\le$$
$$\left[\frac{\left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}\frac{[U_{0max}-U_{0min}]^3}{4}+|N^2|_{max}[U_{0max}-U_{0min}]^2-|N^2|_{min}\frac{[U_{0max}-U_{0min}]^2}{4}}{c_i^2}\right]\int\frac{b|W|^2}{|U_0-c|^2}dz.$$

Therefore,

$$\int\left[\frac{|(bW)'|^2}{b}+k^2b|W|^2\right]dz\le$$
$$[U_{0max}-U_{0min}]^2\left[\frac{\left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}\frac{[U_{0max}-U_{0min}]}{4}+|N^2|_{\max}-\frac{|N^2|_{min}}{4}}{c_i^2}\right]\int\frac{b|W|^2}{|U_0-c|^2}dz. \quad (4.1)$$

4.2 Theorem

If $c_i > 0$ then

$$\int\left[\frac{|(bW)'|^2}{b} + k^2 b|W|^2\right]dz$$

$$\leq \left[\left|b\left(\frac{U_0'}{b}\right)'\right|_{max}(U_{0max} - U_{0min}) + |N^2|_{max}\right]\int\frac{b|W|^2}{|U_0 - c|^2}dz.$$

Proof:

Let us consider the transformation,

$$W = (U_0 - c)F, \tag{4.2}$$

We have

$$\frac{|(bW)'|^2}{b} \geq \left|U_0 - c^2\right|\frac{|(bF)'|^2}{b} - 2|U_0 - c||(bF)'||U_0'||bF| + |U_0'|^2|(bF)^2|. \tag{4.3}$$

Using Cauchy-Schwartz inequality, we get

$$\int|U_0 - c||(bF)'||U_0'||F|dz \leq BE, \tag{4.4}$$

where $B^2 = \int|U_0'|^2 b|F|^2 dz, E^2 = \int|U_0 - c|^2 Qdz,$ and $Q = \frac{|(bF)'|^2}{b} + k^2 b|F|^2$.

Using (4.2), (4.3) and (4.4), we have

$$\int\left[\frac{|(bW)'|^2}{b} + k^2 b|F|^2\right]dz \geq (B - E)^2. \tag{4.5}$$

Multiplying (2.1) by (bW^*), integrating and applying boundary conditions (2.2), we get

$$\int\left[\frac{|(bW)'|^2}{b} + k^2 b|W|^2\right]dz + \int\frac{b\left(\frac{U_0'}{b}\right)'}{(U_0 - c)}b|W|^2 dz - \int\frac{N^2}{(U_0 - c)^2}b|W|^2 dz = 0.$$

Taking real parts, we get,

$$\int\left[\frac{|(bW)'|^2}{b} + k^2 b|W|^2\right]dz + \int\frac{b\left(\frac{U_0'}{b}\right)'(U_0 - c_r)}{|U_0 - c|^2}b|W|^2 dz - \int\frac{N^2\left[(U_0 - c_r)^2 - c_i^2\right]}{|U_0 - c|^4}b|W|^2 dz = 0,$$

$$\int\left[\frac{|(bW)'|^2}{b} + k^2 b|W|^2\right]dz \leq \left|-\int\frac{b\left(\frac{U_0'}{b}\right)'(U_0 - c_r)}{|U_0 - c|^2}b|W|^2 dz + \int\frac{N^2\left[(U_0 - c_r)^2 - c_i^2\right]}{|U_0 - c|^4}b|W|^2 dz\right|.$$

Since $|U_0 - c_r| \leq [U_{0max} - U_{0min}]$ and $\left[(U_0 - c_r)^2 - c_i^2\right] \leq |U_0 - c|^2$,

we have

$$\int\left[\frac{|(bW)'|^2}{b} + k^2 b|W|^2\right]dz \leq \left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}(U_{0max} - U_{0min})\int\frac{b|W|^2}{|U_0 - c|^2}dz + |N^2|_{max}\int\frac{b|W|^2}{|U_0 - c|^2}dz;$$

i.e.,

$$\int\left[\frac{|(bW)'|^2}{b} + k^2 b|W|^2\right]dz \leq \left[\left|b\left(\frac{U_0'}{b}\right)'\right|_{max}(U_{0max} - U_{0min}) + |N^2|_{max}\right]\int\frac{b|W|^2}{|U_0 - c|^2}dz. \tag{4.6}$$

4.3 Theorem

The range of complex phase velocity in $C_r - C_i$ plane is given by

$$\left[c_r - \left(\frac{U_{0max} + U_{0min}}{2}\right)\right]^2 + c_i^2 + J_0\left(1 + \frac{A_1}{c_i^2}\right)^2 c_i^2 \leq \left(\frac{U_{0max} - U_{0min}}{2}\right)^2,$$

where

$$A_1^2 = \frac{(U_{0max} - U_{0min})^2\left[\left|b\left(\frac{U_0'}{b}\right)'\right|_{max}\left(\frac{U_{0max} - U_{0min}}{4}\right) + |N^2|_{max} - \frac{|N^2|_{min}}{4}\right]}{\frac{\pi^2 b_{min}}{D^2 b_{max}} + k^2}.$$

Proof:

Let us consider the transformation,

$$W = (U_0 - c)F,$$
$$bW = (U_0 - c)(bF),$$
$$(bW)' = (U_0 - c)(bF)' + U_0'(bF).$$

We have

$$\frac{|(bW)'|^2}{b} \geq |U_0 - c^2| \frac{|(bF)'|^2}{b} - 2|U_0 - c||(bF)'||U_0'||bF| + |U_0'|^2 |(bF)^2|. \tag{4.7}$$

Using Cauchy-Schwartz inequality, we get

$$\int |U_0 - c||(bF)'||U_0'||F|\,dz$$
$$\leq \left[\int |U_0'|^2 b|F|^2 dz\right]^{\frac{1}{2}} \left[\int |U_0 - c|^2 \frac{|(bF)'|^2}{b} dz\right]^{\frac{1}{2}},$$

$$\int |U_0 - c||(bF)'||U_0'||F|\,dz \leq BE. \tag{4.8}$$

where

$$B^2 = \int |U_0'|^2 b|F|^2 dz, \tag{4.9}$$

$$E^2 = \int |U_0 - c|^2 Q dz, \tag{4.10}$$

and

$$Q = \frac{|(bF)'|^2}{b} + k^2 b|F|^2.$$

Therefore,

$$\int \left[\frac{|(bW)'|^2}{b} + k^2 b|W|^2\right] dz \geq (B - E)^2. \tag{4.11}$$

Using (4.1) and (4.11)

$$(B-E)^2 \leq [U_{0max} - U_{0min}]^2 \left[\frac{\left|b\left(\frac{U_0'}{b}\right)'\right|_{max} \frac{[U_{0max} - U_{0min}]}{4} + |N^2|_{max} - \frac{|N^2|_{min}}{4}}{c_i^2}\right] \times \int b|F|^2\,dz. \tag{4.12}$$

$$E^2 = \int |U_0 - c|^2 Q dz.$$

We know that $|U_0 - c|^2 \geq c_i^2$.

Therefore, we have

$$E^2 \geq c_i^2 \int \left[\frac{|(bF)'|^2}{b} + k^2 b|F|^2\right] dz.$$

Using Rayleigh-Ritz inequality, we get

$$E^2 \geq c_i^2 \left[\frac{\pi^2 b_{min}}{D^2 b_{max}} + k^2\right] \int b|F|^2\,dz,$$

$$\int b|F|^2\,dz \leq \frac{E^2}{c_i^2 \left[\frac{\pi^2 b_{min}}{D^2 b_{max}} + k^2\right]}. \tag{4.13}$$

Substituting (4.13) in (4.12), we get

$$(B-E)^2 \leq [U_{0max} - U_{0min}]^2 \left[\frac{\left|b\left(\frac{U_0'}{b}\right)'\right|_{max} \frac{[U_{0max} - U_{0min}]}{4} + |N^2|_{max} - \frac{|N^2|_{min}}{4}}{c_i^2}\right] \frac{E^2}{c_i^2 \left[\frac{\pi^2 b_{min}}{D^2 b_{max}} + k^2\right]},$$

$$(B-E)^2 \leq \frac{[U_{0max} - U_{0min}]^2}{c_i^2} \left[\left|b\left(\frac{U_0'}{b}\right)'\right|_{max} \frac{[U_{0max} - U_{0min}]}{4} + |N^2|_{max} - \frac{|N^2|_{min}}{4}\right] \frac{E^2}{c_i^2 \left[\frac{\pi^2 b_{min}}{D^2 b_{max}} + k^2\right]}.$$

$$E^2 \left(\frac{B}{E} - 1\right)^2 \leq \frac{A_1^2}{c_i^4} E^2,$$

where

$$A_1^2 = \frac{[U_{0max} - U_{0min}]^2 \left[\left| b\left(\frac{U_0^{'}}{b}\right)^{'} \right|_{max} \frac{[U_{0max} - U_{0min}]}{4} + |N^2|_{max} - \frac{|N^2|_{min}}{4} \right]}{\left[\frac{\pi^2 b_{min}}{D^2 b_{max}} + k^2 \right]}.$$

$$\left(1 - \frac{B}{E}\right)^2 \le \frac{A_1^2}{c_i^4},$$
$$\left(1 - \frac{A_1}{c_i^2}\right) \le \frac{B}{E}, \tag{4.14}$$
$$E^2 \left(1 + \frac{A_1}{c_i^2}\right)^2 \le B^2$$

$$\int N^2 b |F|^2 \, dz \ge \left[\frac{N^2}{\left(U_0^{'}\right)^2} \right]_{min} \int \left|U_0^{'}\right|^2 b |F|^2 \, dz;$$

Let $J_0 = Min \left[\frac{N^2}{\left(U_0^{'}\right)^2} \right]$

i.e.,

$$\int N^2 b |F|^2 \, dz \ge J_0 B^2. \tag{4.15}$$

Substituting (4.15) in (4.14)

$$\int N^2 b |F|^2 \, dz \ge J_0 E^2 \left(1 + \frac{A_1}{c_i^2}\right)^2,$$
$$\int N^2 b |F|^2 \, dz \ge J_0 \left(1 + \frac{A_1}{c_i^2}\right)^2 E^2, \tag{4.16}$$
$$\int N^2 b |F|^2 \, dz \ge J_0 \left(1 + \frac{A_1}{c_i^2}\right)^2 c_i^2 \int Q dz.$$

From (Deng *et. al.* 2008),

$$\left[\left[c_r - \left(\frac{U_{0max} + U_{0min}}{2} \right) \right]^2 + c_i^2 - \left(\frac{U_{0max} - U_{0min}}{2} \right)^2 \right] \int Q dz + \int N^2 b |F|^2 \, dz \le 0.$$

Substituting (4.16) in the above equation, we get

$$\left[c_r - \left(\frac{U_{0max} + U_{0min}}{2} \right) \right]^2 + c_i^2 + J_0 \left(1 + \frac{A_1}{c_i^2}\right)^2 c_i^2 \le \left(\frac{U_{0max} - U_{0min}}{2} \right)^2.$$

Where

$$A_1^2 = \frac{[U_{0max} - U_{0min}]^2 \left[\left| b\left(\frac{U_0^{'}}{b}\right)^{'} \right|_{max} \frac{[U_{0max} - U_{0min}]}{4} + |N^2|_{max} - \frac{|N^2|_{min}}{4} \right]}{\left[\frac{\pi^2 b_{min}}{D^2 b_{max}} + k^2 \right]}.$$

4.4 Theorem

The range of complex phase velocity in $c_r - c_i$ plane is given by

$$\left[c_r - \left(\frac{U_{0max} + U_{0min}}{2} \right) \right]^2 + c_i^2 + \frac{J_0}{\left(1 + \frac{A_2}{c_i}\right)^2} c_i^2 \le \left(\frac{U_{0max} - U_{0min}}{2} \right)^2,$$

where

$$A_2^2=\frac{(U_{0max}-U_{0min})^2\left[\left|b\left(\frac{U_0'}{b}\right)'\right|_{max}\left(\frac{U_{0max}-U_{0min}}{4}\right)+|N^2|_{max}-\frac{|N^2|_{min}}{4}\right]}{|U_0'|_{min}^2}.$$

Proof:

From Theorem 4.3 consider the equation (4.9)

$$B^2=\int\left|U_0'\right|^2 b\left|F\right|^2\,dz,$$

$$B^2\geq\left|U_0'\right|^2\int b\left|F\right|^2\,dz,$$

Therefore,

$$\int b|F|^2\,dz\leq\frac{B^2}{|U_0'|_{min}^2}. \tag{4.17}$$

Substituting (4.17) in (4.12), we get

$$(E-B)^2\leq[U_{0max}-U_{0min}]^2\left[\frac{\left|b\left(\frac{U_0'}{b}\right)'\right|_{max}\frac{[U_{0max}-U_{0min}]}{4}+|N^2|_{max}-\frac{|N^2|_{min}}{4}}{c_i^2}\right]\frac{B^2}{|U_0'|_{min}^2}.$$

$$B^2\left(\frac{E}{B}-1\right)^2\leq\frac{A_2^2}{c_i^2}B^2,$$

$$\left(\frac{E}{B}-1\right)^2\leq\frac{A_2^2}{c_i^2},$$

where

$$A_2^2=\frac{(U_{0max}-U_{0min})^2\left[\left|b\left(\frac{U_0'}{b}\right)'\right|_{max}\left(\frac{U_{0max}-U_{0min}}{4}\right)+|N^2|_{max}-\frac{|N^2|_{min}}{4}\right]}{|U_0'|_{min}^2}.$$

$$\left(\frac{E}{B}-1\right)\leq\frac{A_2}{c_i},$$

$$\left(\frac{E}{B}\right)\leq 1+\frac{A_2}{c_i},$$

$$B^2\geq\frac{E^2}{\left[1+\frac{A_2}{c_i}\right]^2}. \tag{4.18}$$

$$\int N^2 b|F|^2\,dz\geq\left[\frac{N^2}{\left(U_0'\right)^2}\right]_{min}\int|U_0'|^2\,b\,|F|^2\,dz;$$

i.e.,

$$\int N^2 b|F|^2\,dz\geq J_0B^2. \tag{4.19}$$

Substituting (4.18) in (4.19)

$$\int N^2 b|F|^2\,dz\geq\frac{E^2J_0}{\left[1+\frac{A_2}{c_i}\right]^2},$$

$$\int N^2 b|F|^2\,dz\geq\frac{J_0}{\left[1+\frac{A_2}{c_i}\right]^2}c_i^2\int Q\,dz,$$

From (Deng *et. al.* 2008),

$$\left[\left[c_r - \left(\frac{U_{0max}+U_{0min}}{2}\right)\right]^2 + c_i^2 - \left(\frac{U_{0max}-U_{0min}}{2}\right)^2\right]\int Qdz + \int N^2 b\,|F|^2\,dz \le 0,$$

$$\left[c_r - \left(\frac{U_{0max}+U_{0min}}{2}\right)\right]^2 + c_i^2 + \frac{J_0}{\left[1+\frac{A_2}{c_i}\right]^2}c_i^2 \le \left(\frac{U_{0max}-U_{0min}}{2}\right)^2,$$

where

$$A_2^2 = \frac{(U_{0max}-U_{0min})^2\left[\left|b\left(\frac{U_0^{'}}{b}\right)^{'}\right|_{max}\left(\frac{U_{0max}-U_{0min}}{4}\right)+|N^2|_{max}-\frac{|N^2|_{min}}{4}\right]}{|U_0^{'}|_{min}^2}.$$

5. Conclusions

In this article, we have obtained the bounds for the complex part of the phase velocity . It depends on number of various parameters. We have also obtained an instability region depends on various parameters like stratification parameter, vorticity variation and wave number.

References

Deng, J., Pratt, L., Howard, L., Jones, C. (2008). "On stratified shear flow in sea straits of arbitrary cross section", Stud. Appl. Math., **111,** 409–434. (https://doi.org/10.1111/1467-9590.t01-1-00040)

Ganesh, V. (2010). New Upper Bound for the growth rate and a modified instability region for the extended Taylor-Goldstein problem of hydrodynamic stability, Ind. J. Maths, **52,** 415–427. (www.amsallahabad.org /pdf/ijm522.pdf)

Ganesh, V., Subbiah, M. (2009). "On upper bounds for the growth rate in the extended Taylor-Goldstein problem of hydrodynamic stability", Proc. Indian Acad. Sci., **119,** 119–135. (https://doi.org/10.1007/s12044-009-0012-5)

Pratt, L., Deese, H. E., Murray, S. P. (2000). Johns. W., "Continuous dynamical modes in sea straits having arbitrary cross section with applications to the Bobal Mandab", J. Phys. Oceanogo., **30,** 2515–2534. (https://doi.org/10.1175/1520-0485(2000)030<2515:CDMISH>2.0.CO;2)

Reddy, V. R., Subbiah, M. (2014). "Long wave instability of shear flows in sea straits", Proc. 59th Conf. Ind. Soc. Theo. and Appl. Mech., 1–10. (https://doi.org/istam.iitkgp.ac.in)

Reenapriya, K., Ganesh, V. (2015). On the instability region for the extended taylor-Goldstein problem of hydrodynamic stability, Appl. Maths. Sci., **9,** 2265–2272. (https://doi.org/10.12988/ams.2015.53207)

Sridevi, S., Ganesh, V. (2015). On the Bounds for the Complex Wave Velocity for Growing Perturbations, Applied Mathematical Sciences, **9,** 2245–2253, (https://doi.org/10.12988/ams.2015.53205)

Sridevi. S. and Ganesh, V. (2016). On the stability of shear flows with bottom topography, International Journal of Mathematical Analysis, **10,** 651–659. (https://doi.org/10.12988/ijma.2016.6225)

Sridevi, S. and Ganesh, V. (2018). On the Long wave stability of Shear flows in a sea straits of arbitrary cross section, Int. J. Appl. and Comp. Maths,. (https://doi.org/10.1007/s40819-017-0439-9)

Sridevi, S. Hua-Shu, D. and Ganesh, V. (2017). A Unified Instability Region for the extended Taylor -Goldstein Problem of Hydrodynamic Stability, Adv. in Appl. Maths and Mech., **9,** 1404–1419. (https://doi.org/10.4208/aamm.OA-2016-0022)

Subbiah, M., Ganesh, V. (2008). "Bounds on the phase speed and growth rate of extended Taylor-Goldstein problem", Fluid Dynamic Research, **40,** 364–377. (https://doi.org/10.1016/j.fluiddyn.2007.11.001)