Math. J. Interdiscip. Sci.Vol. 7, No. 2, March 2019, pp. 121–123

CHITKARA

Mathematical Journal of Interdisciplinary Sciences

Journal homepage: https://mjis.chitkara.edu.in/

On Some Special Property of The Farey Sequence

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ARTICLE INFORMATION

Received: 14 November, 2018
Revised: 17 December, 2018
Accepted: 11 January, 2019

Published online: March 6, 2019

Keywords:
Farey sequence, Palindrome, Euler's Totient function.

DOI: https://doi.org/1015415/mjis.2019.72016

ABSTRACT

In this paper, some special property of the Farey sequence is discussed. We prove in each term of the Farey sequence, the sum of elements in the denominator is two times of the sum of elements in the numerator. We also prove that the Farey sequence contains a palindrome structure.

1. Introduction

Farey sequence is named after British geologist John Farey, Sr., who published a result in Philosophical Magazine in 1816 about this sequence without giving a proof. Later, Cauchy proved the result conjectured by Farey. Though, Charles Haros proved a similar result in 1802 which were not known to both Farey and Cauchy. Later, Farey sequence appeared in many different areas of mathematics, including number theory, topology, geometry, see, (Hardy and Wright 1979), (Cobeli and Zaharescu 2003), (Ainsworth, Dawson, Pianta and Warwick 2012). Farey sequence is also related to Ford's circle and Riemann hypothesis, (Kanemitsu and Yoshimoto 1996). Farey sequence (Hardy and Wright 1979), F_n in the interval [0, 1] is defined as a sequence of ascending rational numbers in reduced form starting at $\frac{0}{1}$ and ending at $\frac{1}{1}$ such that the elements in the nth term, denominator are less than or equal to n. First few terms of Farey sequence are listed as follows,

$$F_1 = \frac{0}{1}, \frac{1}{1},$$

$$F_2 = \frac{0}{1}, \frac{1}{2}, \frac{1}{1},$$

$$F_3 = \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1},$$

$$F_4 = \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1},$$

$$F_5 = \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1},$$

For all the terms of the Farey sequence given above, one can note that the summation of the elements in the denominator is always two times of the summation of the elements in the numerator. In F_1, sum of numerators is 1 and sum of denominators is 2. In F_2, sum of numerators is 2 and sum of denominators is 4. Likewise in F_3, F_4, F_5, sum of denominators is always two times the sum of numerators. So, one may ask if this result holds for all F_n or not. In this paper, we will show that in general this result is true. In the second section of this article, we introduce this special property of Farey sequence. A palindrome is a word or sequence if it reads same from forward and backward direction. Palindrome appears in many different areas. In the final section, we show that denominators of each fraction in F_n has a palindrome structure for all n.

2. A Special Property of The Farey Sequence

In this section, we prove our first result on Farey sequence. We show the summation of the elements in the denominator is two times of the summation of elements in the numerator for all positive integer n.

2.1 **Lemma [1]** The length of each term of Farey sequence F_n is given by the following recurring formula:

$$|F_n| = |F_{n-1}| + \varphi(n).$$

ISSN No.: 2278-9561 (Print) ISSN No.: 2278-957X(Online) Registration No. : CHAENG/2013/49583 Periodicity: Bi-Annually

2.2. **Remark**. From the Lemma 2.1, we get the information that number of new elements appears in F_n compare to F_{n-1} is $\varphi(n)$. The numbers $1\leq k < n$ which are co-prime to n, appears to the numerator in the new terms of F_n and the denominator of each new terms in F_n is always n.

2.3. **Lemma [2].** For all integer $n \geq 0$ and $k \geq 0$,

$$\sum_{(n,k)=1} k = \frac{n\varphi(n)}{2}.$$

Proof. If gcd(n, k)=1, then gcd(n, n−k)=1. Note that, k can not be equal to $n-k$, otherwise, $\gcd(k, n)$ will not be 1. The number of elements co-prime to n is $\varphi(n)$. So pairing k and $n-k$, we get the total sum is $\frac{n\varphi(n)}{2}$.

2.4. Theorem. In the Farey sequence F_n, the summation of the elements in the denominator is two times of the summation of elements in the numerator for all positive integer n.

Proof. We use induction to prove this result. Let N_n denotes the summation of the elements in the numerator of the nth term of Farey sequence F_n and D_n denotes the summation of elements in the denominator of nth term of Farey sequence.

For n = 1, $N_1 = 1$ and $D_1 = 2$.

For n = 1, $D_1 = 2N_1$. Now assume the result holds for n − 1. We prove that the result holds for all n, we have,

$$\begin{aligned} N_n &= N_{n-1} + \textstyle\sum_{\gcd(n,k)=1} k, \text{ using remark 2.2} \\ &= N_{n-1} + \frac{n\varphi(n)}{2}, \text{using lemma 2.3} \\ D_n &= D_{n-1} + n\varphi(n), \text{using remark 2.2} \\ &= 2N_{n-1} + n\varphi(n), \text{using induction.} \\ &= 2\left(N_{n-1} + \frac{n\varphi(n)}{2}\right) \\ &= 2N_n \end{aligned}$$

So, by induction our result follows.

3. Palindromic Structure in The Farey Sequence

In this final section, we show that there is a palindrome structure in denominators of F_n for all n. Thus from the Farey sequence we can get palindrome numbers.

3.1 Definition. A word or sequence is called a Palindrome if it reads same from forward and backward direction. Palindrome appears in many different areas. In names, numbers or musical notes appearance of palindromes is noticeable. For example, english word BOB, MADAM, MOM and numbers 101, 1001, 12321, are examples of palindromes.

3.2. Definition. Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ in F_n are called Farey neighbours if $\frac{a}{b} < \frac{c}{d}$ and there is no fraction in F_n between them.

The following lemma gives a condition for two fraction to be Farey neighbour.

3.3 Lemma: [1]: Let, $\frac{0}{1} \leq \frac{a}{b} < \frac{c}{d} \leq \frac{1}{1}$ then $\frac{a}{b}, \frac{c}{d} \in F_n$ are neighbours in the Farey sequence if and only if $bc - ad = 1$.

3.4 Corollary: If $\frac{0}{1} \leq \ldots < \frac{a_1}{b_1} < \frac{a_2}{b_2} < \ldots \leq \frac{1}{2}$ are Farey neighbours, then $\frac{1}{2} \leq \ldots < \frac{b_2 - a_2}{b_2} < \frac{b_1 - a_1}{b_1} < \ldots \leq \frac{1}{1}$ are also Farey neighbours

Proof. Using lemma 3.3, on the given first Farey neighbours, we have,

$$b_1 a_2 - a_1 b_2 = 1 \tag{3.1}$$

So, $b_2(b_1 - a_1) - b_1(b_2 - a_2) = b_1 a_2 - a_1 b_2 = 1$

Now, $1 - \frac{a_1}{b_1} \geq \frac{1}{2}$ as $\frac{0}{1} \leq \ldots < \frac{a_1}{b_1} \leq \frac{1}{2}$. Thus, $\frac{b_1 - a_1}{b_1} \geq \frac{1}{2}$.

Similarly, $\frac{1}{2} \leq \frac{b_2 - a_2}{b_2} \leq \frac{1}{1}$. Thus, our corollary follows.

3.5. Theorem. Denominators of each fraction in F_n for all n in the Farey sequence is a palindrome.

Proof. We prove this theorem by induction.

In F_1, denominators are 1,1, which is a palindrome sequence. In F_2, denominators are 1,2,1, which is also a palindrome sequence.

Now, suppose that denominators in F_{n-1} are in a palindrome sequence. We need to show that denominators in F_n is also in a palindrome sequence. Using corollary 3.4 and palindrome structure of denominators in F_{n-1}, we can write F_{n-1} in the following form,

$$F_{n-1} = \left\{\frac{0}{1}, \ldots, \frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{1}{2}, \ldots, \frac{s_2 - r_2}{s_2}, \frac{s_1 - r_1}{s_1}, \ldots, \frac{1}{1}\right\}.$$

Suppose in the next term F_n, a new term appears between $\frac{r_1}{s_1}$, and $\frac{r_2}{s_2}$.

Let, $r_1 + r_2 = k$. We have following relations by using remark 2.2, $s_1 + s_2 = n$ and gcd(n, k)=1.

Now, $(s_1 - r_1) + (s_2 - r_2) = n - k$.

Thus we write F_n in terms of n, k as follows:

$$\left\{\frac{0}{1}, \ldots, \frac{r_1}{s_1}, \frac{k}{n}, \frac{r_2}{s_2}, \ldots, \frac{1}{2}, \ldots, \frac{s_2 - r_2}{s_2}, \frac{n-k}{n}, \frac{s_1 - r_1}{s_1}, \ldots, \frac{1}{1}\right\}$$

As $\frac{r_1}{s_1} < \frac{\mathrm{k}}{\mathrm{n}} < \frac{r_2}{s_2}$ are Farey neighbours, we have,

$$ks_1 - nr_1 = 1 \text{ and } nr_2 - ks_2 = 1 \tag{3.2}$$

Note that,

$$\mathrm{n}(s_1 - r_1) - (n-k)s_1 = ks_1 - nr_1 = 1,$$

$$(n-k)s_2 - n(s_2 - r_2) = nr_2 - ks_2 = 1.$$

So, $\frac{s_2 - r_2}{s_2} < \frac{\mathrm{n-k}}{\mathrm{n}} < \frac{s_1 - r_1}{s_1}$ are Farey neighbours. Thus, we have proved that denominators in F_{n} are palindrome. Our desired result follows from induction.

3.6 Remark. By length of a palindrome, we mean the number of element appearing in the palindrome. From the lemma 2.1, it is clear that palindromic length of denominator palindrome in F_{n} is $|\mathrm{F}_{\mathrm{n-1}}| + \varphi(\mathrm{n})$. Thus using the Farey sequence one can get very large palindrome numbers.

Acknowledgements

During this work the author was supported by DST-INSPIRE Fellowship, Govt. of India and author is thankful to Sumit Mishra of Emory University for introducing Farey sequence to him. The author is also thankful to Krishnadhan Sarkar for his help in type setting in the word format.

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