Commutativity of Prime Ring with Orthogonal Symmetric Biderivations

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ABSTRACT

The intention of the present research article is to generalize the performance of prime rings (commutativity) with certain algebraic identities using Jordan ideals. Familiar results characterizing commutativity of prime ring with orthogonal biderivations have been discussed here with Jordan ideals. Whenever some biderivations of prime ring satisfying certain commutator relations \[ [B_1(u, v), B_2(v, w)] = [u, w], \quad B_1(v, w)B_1(u, v) - B_2(w, u)B_2(v, w) = [u, v], \] for all \( u, v, w \in R \) then that ring is commutative.

1. Introduction

More than a few authors, investigated the structure prime ring & semiprime rings (commutativity) accepting the derivations, generalized derivations etc. The notion of derivations of prime rings was originated by (Posner 1957), jordan derivations of prime rings was originated by (uczak 1975). These derivations was extended by (Bell and Daif 1995) for commutativity of prime rings. Later on (Bresar 1993) used centralizing concept using derivations. These generalizations was done in the article derivations using semiprime rings with results are commutative by (Daif 1998). The concept of symmetric biderivations on prime and semiprime rings was introduced by (Vukman 1989). The notation and terminology in this paper follows (Vukman 1990 and Oukhtite 2011). Many authors have their contribution to orthogonality of derivations on semiprime as well as prime rings. The idea of orthogonality of derivations on semiprime as well as prime rings was developed by (Vukman and Bresar 1989). (Argac 2004) studied orthogonality conditions for generalized derivations. (Ashraf 2010) obtained the orthogonality conditions for a pair of derivations in gamma rings. with their results (Jaya Subha Reddy et. al. 2016) obtained the essential and sufficient conditions of biderivations to be orthogonal. (Oukhtite et. al. 2014) proved the commutativity results of prime rings with derivations using jordan ideals. In this current study it was extended the results of commutativity of prime rings with orthogonal biderivations using Jordan ideals.

2. Preliminaries

In each part of this article all rings assumed to be associative and possesses an identity. As a well-known the commutator \( (uv - vu) \) will be symbolized as \( [u, v] \). We are wellknown that \( R \) is a prime ring if \( uR = 0 \Rightarrow u = 0 \) or \( v = 0 \) and is semiprime if \( uRu = 0 \Rightarrow u = 0 \). If \( D(u) = D(u) v + vD(u), \) for any \( u, v \in R \) then we call this additive map \( D: R \rightarrow R \) is a derivation. We Defined, biadditive mapping \( B(., .): R \times R \rightarrow R \) as a symmetric biderivation if \( B(\alpha u, \beta v) = B(\alpha v, \beta u) = \alpha \beta B(u, v) \). Clearly, in next case also \( B(u, vr) = B(u, v)r + vB(u, r) \), for every \( u, v, r \in R \). Any pair \( (d, g) \) of derivations
are orthogonal if d(u)Rg(v) = 0 = g(v)Rd(u) for any u, v ∈ R (Vukman and Bresar 1989). Likewise, any pair (B, D) of biderivations are said to be orthogonal if B(u, v)RD(v, r) = (0) = D(v, r)RB(u, v) for all u, v, r ∈ R. If u · x ∈ J, for any u ∈ J, x ∈ R, then we say J is a Jordan ideal of R. Note that B(x) means B(x, m) means for some m ∈ J.

In the entire paper R act as a prime ring with 2-torsion free & J ≠ 0 is a jordan ideal of R

Following known results to the readers

Res 1: If [u, u²] = 0 for any u ∈ J, then is in center of R.

Res 2: If an additive subgroup is a subset of Z(R), then R is commutative ring.

Res 3: A semiprime ring R of characteristic not two, a pair of biderivations B_1 and B_2 satisfies the condition B_1(B_2(u, v) − u) = 0, for every pairs u, v ∈ J, then orthogonality of B_1 and B_2 are satisfied, also B_1 = 0 or B_2 = 0.

Proof: Consider B_2(u, v) ≠ 0

We have B_1B_2(u, v) ≠ 0 , for any u, v ∈ J

(1)

By using lemma 2.1 B_1, B_2 are orthogonal, that is

B_1(u, m)B_2(v, w) + B_2(u, w)B_1(v, m) = 0, for any m ∈ J .

(2)

From lemma 2.1, B_1 and B_2 are orthogonal.

One can see that the equation (9) is same as compared with equation (2) so using the above lemma, we conclude B_1(u, v) = 0.

Lemma 2.4

Any two biderivations B_1 and B_2 satisfies the condition B_1(B_2(u, v) − u) = 0, for every pairs u, v ∈ J, then orthogonality of B_1 and B_2 are satisfied, also B_1 = 0 or B_2 = 0.

Proof: Consider for every u, v ∈ J, B_1(B_2(u, v) − u) = 0.

(7)

If R is commutative, substitute u by u² in equation (7), we get

B_1(u, m)B_2(u, v) = 0.

(8)

Already a well known result, from the definition, a prime ring itself an integral domain, so the equation (8) reduces to

B_1(u, m) = 0 or B_1(u, v) = 0.

If B_1(u, v) ≠ 0 then B_1(u, m) = 0 and let R is non commutative and B_1(u, v) ≠ 0. Put u by up in the equation (7), where p ∈ J, find that

B_1(u, w)B_2(p, v) + B_1(u, v)B_2(p, w) = 0.

(9)

From lemma 2.1, B_1 and B_2 are orthogonal.
Replacing $w$ by $[s, t]w$ for some $[s, t]w \in J$, where $s, t \in R$ in the equation (11), to get.

\[ B_i(u, v)[s, t]wB_i(v, m) = [s, t]w[u, m] \]  

(12)

Left multiplication of (11) by $[s, t]$, to get

\[ [s, t]B_i(u, v)wB_i(v, m) = [s, t]w[u, m] . \]  

(13)

From equation (12) and equation (13), we get

\[ B_i(u, v)[s, t]wB_i(v, m) = 0 . \]  

Since $B_2(v, m) = 0$, the primeness of $R$ implies that

\[ [B_i(u, v), [s, t]] = 0 . \]  

(14)

So $B_i(u, v)$ is commuting, then Bresar (Bresar 1993), gives that $R$ is commutative and equation (10) becomes

\[ B_i(u, v)B_j(v, w) = 0 . \]  

And also $B_i$ and $B_j$ are orthogonal biderivations.

(15)

Because of $B_i(v, w) = 0$, leads to $B_i(u, v) = 0$

Using (10), we conclude that $[u, w] = 0$, therefore $R$ is commutative.

3. Main Theorems

**Theorem 3.1**

Any two biderivations $B_i$ and $B_j$ satisfies the condition

\[ [B_i(u, v), B_j(v, w)] = [u, w] , \]  

for every $u, v, w \in J$ then $B_i$ and $B_j$ are orthogonal biderivations. If

\[ B_i = 0 \text{ or } B_j = 0 , \]  

then the given condition becomes $[u, w] = 0$, for any $u, w \in J$, so $R$ is commutative.

**Proof:** If $B_i = 0$ or $B_j = 0$, then $R$ is commutative.

Next consider $B_i$ and $B_j$ are nonzero biderivations implies

\[ [B_i(u, v), B_j(v, w)] = [u, w] , \]  

for all $u, v, w \in J$.  

By replacing $w$ with $wm$, $m \in J$ in the Eq. (16), to get

\[ B_i(v, w)][B_i(u, v), m] + [B_i(u, v), w]B_i(v, m) = 0 . \]  

(17)

By replacing $w$ with $w[p, q]$ in the equation (17), to obtain

\[ B_i(v, w)[s, pq][B_i(u, v), m] + [B_i(u, v), w][s, pq]B_i(v, m) = 0 , \]  

for every $p, q \in J, s \in R$.

By replacing $w$ with $tw$ for some $t \in R$, in (18), we get

\[ B_i(v, t)w[s, pq][B_i(u, v), m] + [B_i(u, v), t]w[s, pq]B_i(v, m) = 0 . \]  

(19)

By replacing $t$ with $B_i(u, v)t$ in the equation (19), we get

\[ B_i(u, v)B_i(v, w)tw[s, pq][B_i(u, v), m] = 0 . \]  

(20)

Using the primeness of $R$, either $B_iB_j = 0$ which implies that orthogonality of biderivations $B_i$ and $B_j$ is satisfied using lemma 2.1. Now from the Res 3 and using lemma 2.2, either $B_j = 0$ or $B_j = 0$ then $R$ is commutative. Otherwise $[s, pq][B_i(u, v), m] = 0$. In such case also is commutative.

**Corollary 3.1**

Any two biderivations $B_i(u, v)$ and $B_j(v, w)$ satisfies the condition

\[ [B_i(u, v), B_j(v, w)] = 0 , \]  

for every $u, v, w \in J$ then $R$ is commutative.

**Proof:** Given condition that

\[ [B_i(u, v), B_j(v, w)] = 0 , \]  

for every $u, v, w \in J$.  

By replacing $u$ with $um$, in the equation (21), we get

\[ [B_i(u, v), B_j(v, m)] = 0 . \]  

(22)

We observed that the equation (22) and equation (17) are identical, continuing the procedure as we done in the theorem 3.1, it is clear that $R$ is commutative.

3.2

For any three nonzero biderivations $B_i, B_j$ and $B_k$ of $R$ satisfies one of the conditions

(i) $B_i(v, w)B_j(w, u) = B_j(w, u)B_i(v, w)$,

(ii) $B_i(v, w)B_j(w, u) - B_j(w, u)B_i(v, w) = [u, v],$, for every $u, v, w \in J$ then $R$ is commutative and $B_i = B_j$.

**Proof:** (i) Let us consider the condition $B_i(v, w)B_j(w, u) - B_j(w, u)B_i(v, w) = 0$.

(23)

Replacing $um^2$ in place of $u$ in the Eq. (23), to get

\[ B_i(v, w)B_j(w, u)m^2 + B_j(w, u)B_i(v, w) - uB_j(w, u)m^2B_i(v, w) = 0 . \]  

(24)

Replacing $u$ by $u[p, q]$ in the equation (24), we get

\[ B_i(v, w)[s, pq]B_j(v, w)m^2 + B_j(v, w)[s, pq]B_i(v, w)m^2 = 0 . \]  

(25)

Replacing $u$ with $tu^2$ in the equation (25), to get

\[ B_i(v, w)u^2[s, pq]B_j(v, w)m^2 + B_j(v, w)u^2[s, pq]B_i(v, w)m^2 = 0 . \]  

(26)

Replacing $t$ with $B_i(v, w)t$ in equation (26), to get

\[ B_i(v, w)B_j(v, w)Ru[s, pq]B_j(v, w)m^2 = 0 . \]  

(27)

It is clear that (27) and (20) are identical, hence we conclude that $R$ is commutative then equation (23) becomes
\( B_3(v, w) R [B_1(w, u) - B_2(w, u)] = 0 \) \hspace{1cm} (28)

Since \( B_3(q, r) \) is non zero, \( B_1(w, u) = B_2(w, u) \).

(ii) Consider \( B_1 \) and \( B_2 \) are non zero biderivations such that
\[ B_3(v, w) B_1(w, u) - B_2(w, u) B_3(v, w) = [u, v]. \] \hspace{1cm} (29)

Replacing \( u \) with \( um^2 \) in the equation (29), we get
\[ B_1(v, w) B_1(w, um^2) - B_2(r, um^2) B_1(v, w) = [um^2, v] \]
\[ B_2(w, u) [B_1(v, w), m^2] + [B_1(v, w), u] B_1(w, m^2) = 0 . \] \hspace{1cm} (30)

It is clear that the equation (30) and the equation (24) are identical, proceeding the same procedure as in (i) it is clear that \( R \) is commutative, then condition (ii) becomes
\[ B_3(v, w) R [B_1(w, u) - B_2(w, u)] = 0 \]
Since \( B_3(v, w) \neq 0 \), leads to \( B_1 = B_2 \). \hspace{1cm} (31)

References


