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Controllability and Observability of Kronecker Product Sylvester System on Time Scales

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Abstract
Main objective in this paper is to present the necessary and sufficient conditions for complete controllability, complete observability associated with kronecker product Sylvester system on time scales.

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1. INTRODUCTION

The importance of kronecker product Sylvester systems on time scales is an interesting area of current research. Which arise in number of areas of control engineering problems, dynamical systems, and feedback systems are well known. There are many results from differential equations that carry over quite naturally and easily to difference equations, while others have a completely different structure for their continuous counterparts. The study of Sylvester system on time scales sheds new light on the discrepancies between continuous and discrete Sylvester systems. It is also prevents one from proving a result twice on for continuous and once for discrete systems. The general idea, which the main goal of Bhoner and Peterson's introductory text [1] is to prove a result for a first order differential equation when the domain of the unknown function is so-called timescale.

In this section, we shall be concerned with the first order Δ -differentiable kronecker product dynamical system represented by

$$(X(\mathrm{t})\otimes \mathrm{Y(t)})^{\Delta}=(\mathrm{A(t)}\oplus \mathrm{C(t)})(\mathrm{X(t)}\otimes \mathrm{Y(t)})+(\mathrm{X}(\sigma(\mathrm{t})\oplus \mathrm{Y}(\sigma(\mathrm{t}))(\mathrm{B(t)}\oplus \mathrm{D(t)})+\mathrm{F(t)}\oplus \mathrm{G(t)}(\mathrm{U(t)}\oplus \mathrm{V(t)}),$$

$$(X(t_0)\otimes Y(t_0)=(P_0 \otimes \mathrm{Q}_0) \tag{1.1}$$

$$(R(t)\otimes S(t))=(K(t)\otimes L(t))(X(t)\otimes Y(t)) \tag{1.2}$$

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Where A(t), B(t), C(t), D(t), X(t) and Y(t) are square matrices of order n on J=$[t_0, t_1]$, R(t), S(t), K(t) and L(t) are all matrices of order p×n and F(t), G(t) are matrices of order n×m and U(t), V(t) are control matrices of order n×m Here we assume that the matrices A(t), B(t), C(t), D(t) are rd-continuous on closed interval J. Many others {[2], [6], [7]} obtained the controllability, observability criteria for continuous systems .

This paper is well organized as follows: In Section2 we present the general solution of (1.1) in terms of two fundamental matrix solutions of the systems

$$(X(t)\otimes Y(t))^{\Delta} = (A(t)\oplus C(t))(X(t)\otimes Y(t)) \text{ and}$$

$$(X(t)\otimes Y(t))^{\Delta} = (B(t)\oplus D(t))^{*}(X(\sigma(t)\otimes Y(\sigma(t))$$

(*denotes the transpose of a matrix) on time scales and some basic results on kronecker products and timescales are also presented in this section. In Section3 we establish the necessary and sufficient conditions for complete controllability and complete observability under certain smoothness conditions.

2. KRONECKER PRODUCT SYLVESTER SYSTEMS

In this section we present some basic definitions, notations and results which are useful for later discussion.

Definition 2.1: [3] If $P, Q \in C^{n\times n}$ are two square matrices of order 'n' then their Kronecker product(or direct product or tensor product) is denoted by $P\otimes Q \in C^{n^2\times n^2}$ is defined to be partition matrix

$$P\otimes Q = \begin{bmatrix} p_{11}Q & p_{12}Q & \cdots & p_{1n}Q \\ p_{21}Q & p_{22}Q & \cdots & p_{2n}Q \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1}Q & p_{n2}Q & \cdots & p_{nn}Q \end{bmatrix}$$

We shall make use of vector valued function denoted by Vec P of a matrix $P = \{p_{ij}\} \in C^{n\times n}$ defined by

$$\hat{P} = VecP = \begin{bmatrix} P_{.1} \\ P_{.2} \\ \vdots \\ \vdots \\ P_{.n} \end{bmatrix} \quad \text{where} \quad \boldsymbol{P}_{.j} = \begin{bmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ \vdots \\ p_{nj} \end{bmatrix} \quad 1 \le j \le n$$

it is clear that VecP is of order n^2.

The Kronecker product has the following properties[3]

1. $(P \otimes Q)^* = P^* \otimes Q^*$ (P^* denotes the transpose of P)
2. $(P \otimes Q)^{-1} = P^{-1} \otimes Q^{-1}$
3. The mixed product rule $((P \otimes Q)((M \otimes N) = (PM \otimes QN))$. This rule holds good, provided the dimension of the matrices are such that the various expressions exist.
4. If P(t) and Q(t) are matrices, then $(P \otimes Q)' = P' \otimes Q'$ ($' = d/dt$)
5. $Vec(PYQ) = (Q^* \otimes P)Vec\ Y$
6. If P and Q are matrices both of order n×n then

   (i) $Vec(PX) = (I_n \otimes P)VecX$ (ii) $Vec(XP) = (P^* \otimes I_n)VecX$

Now we introduce some basic definitions and results on time scales T[1][5] needed in our subsequent discussion.

A Timescale T is a closed subset of R; and examples of time scales include N; Z; R, Fuzzy sets etc. The set $Q = \{t \in R / Q, 0 \le t \le 1\}$ are not time scales. Time scales need not necessarily be connected. In order to overcome this deficiency, we introduce the notion of jump operators. Forward (backward) jump operator $\sigma(t)$of t for t < sup T (respectively $\rho(t)$ at t for t >inf T) is given by $\sigma(t) = \inf\{s \in T : s > t\}$, ($\rho(t) = \sup\{s \in T : s < t\}$), for all $t \in T$. The graininess function $\mu : T \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$. Throughout we assume that T has a topology that it inherits from the standard topology on the real number R. The jump operators σ and ρ allow the classification of points in a time scale in the way: If $\sigma(t) > t$, then the point t is called right scattered ; while if $\rho(t) < t$, then t is termed left scattered. If t < sup T and $\sigma(t) = t$, then the point ' t' is called right dense: while if t > inf T and $\rho(t) = t$, then we say 't' is left-dense. We say that $f : T \to R$ is rd-continuous provided f is continuous at each right-dense point of T and has a finite left-sided limit at each left-dense point of T and will be denoted by Crd.

A function f : T → T is said to be differentiable at $t \in T^k = \{T \setminus (\rho(t)\max(T), \max t)\}$ if $\lim_{\sigma(t)\to s} \dfrac{f((\sigma(t) - f(s))}{\sigma(t) - s}$ where $s \in T - \{\sigma(t)\}$ exist and is said to be differentiable on T provided it is differentiable for each $t \in T^k$. A function F : T → T, with $F^{\Delta}(t) = f(t)$ for all $t \in T^k$ is said to be integrable, if $\int_s^t f(\tau)\Delta\tau = F(t) - F(s)$ where F is anti derivative of f and for all s, t ∈T. Let f: T → T, and if T=R and a, b ∈ T, then $f^{\Delta}(t) = f'(t)$ and $\int_a^b f(t)dt = \int_a^b f(t)\Delta t$.

If T = Z, then $f^{\Delta}(t) = \Delta f(t) = f(t + 1) - f(t)$ and

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=a}^{b-1} f(k) & if \quad a<b \\ 0 & if \quad a=b \\ \sum_{k=b}^{a-1} f(k) & if \quad a<b \end{cases}$$

If f is Δ-differentiable, then f is continuous. Also if t is right scattered and f is continuous at t then

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}$$

In this section, we present the general solution of the Sylvester system on time scale (1.1) in terms of the fundamental matrix solution of $(X(t)\otimes Y(t))^{\Delta}=(A(t)\oplus C(t))(X(t)\otimes Y(t))$ and

$$(X(t)\otimes Y(t))^{\Delta}=(B(t)\oplus D(t))^{*}(X(\sigma(t)\otimes Y(\sigma(t))$$

Theorem 2.1: If $(Y_1(t)\otimes Z_1(t))$ and $(Y_2^{*}(t)\otimes Z_2^{*}(t))$ are fundamental matrix solutions of

$$(X(t)\otimes Y(t))^{\Delta}=(A(t)\oplus C(t))(X(t)\otimes Y(t)) \text{and}$$

$(X(t)\otimes Y(t))^{\Delta}=(B(t)\oplus D(t))^{*}(X(\sigma(t)\otimes Y(\sigma(t))$ respectively, then any solution of the homogeneous kronecker product Sylvesters system

$$(X(t)\otimes Y(t))^{\Delta}=(A(t)\oplus C(t))(X(t)\otimes Y(t))+(X(\sigma(t)\otimes Y(\sigma(t))(B(t)\oplus D(t))$$

is of the form $(X(t)\otimes Y(t)) =(Y_1(t)\otimes Z_1(t))\,(\zeta_1\otimes\zeta_2)\,(Y_2^{*}(t)\otimes Z_2^{*}(t))$, where ζ_1,ζ_2 are constant square matrices of order n and $Y_1(t)$, $Z_1(t)$, $Y_2(t)$, and $Z_2(t)$ are fundamental matrix solutions of

$$X^{\Delta}(t)=(A(t)(X(t),\ Y^{\Delta}(t)=(C(t)Y(t),\ X^{\Delta}(t)=(B^{*}(t)(X(\sigma(t)),$$
$$Y^{\Delta}(t)=(D^{*}(t)(Y(\sigma(t))$$

respectively.

Theorem2.2: Any solution of (1.1) is of the form

$$(X(t)\otimes Y(t)) =(Y_1(t)\otimes Z_1(t)) \quad (\zeta_1\otimes\zeta_2) \quad (Y_2^{*}(t)\otimes Z_2^{*}(t))+(\bar{X}(t)\otimes \bar{Y}(t))$$

where $(\bar{X}(t)\otimes\bar{Y}(t)$ is a particular solution of(1.1).

Theorem2.3: A particular solution)of (1.1) is given by

$$(\bar{X}(t)\otimes\bar{Y}(t))=(Y_1(t)\otimes Z_1(t))\left[\int_{t_0}^{t}((Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1}(I\otimes F(s))\right.$$
$$\left.\times(U(s)\otimes I(s))(Y_2^*(s)\otimes Z_1^*(s))^{-1}\Delta s\right](Y_2^*(t)\otimes Z_2^*(t))$$

Theorem 2.4: Any solution (X(t)$\otimes$Y(t)) of the initial value problem (1.1) satisfying $(X(t_0)\otimes Y(t_0)=P_0\otimes Q_0$ is given by

$$(X(t)\otimes Y(t)=\phi(t,t_0)P_0\otimes Q_0\psi^*(t,t_0)+\phi(t,t_0)$$
$$\times\left[\int_{t_0}^{t}(\phi(t_0,\sigma(s))(I\otimes F(s))((U(s)\otimes I(s))\psi^*(s,t_0)\Delta s\right]\times\psi^*(t_0,t) \tag{2.2}$$

where $$\phi(t,\sigma(s))=(Y_1(t)\otimes Z_1(t))(Y_1(\sigma(s))\otimes Z_1(\sigma(s)))^{-1} \tag{2.3}$$

and $$\psi^*(s,t)=(Y_2^*((s))\otimes Z_1^*((s)))^{-1}(Y_2^*((t))\otimes Z_1^*((t)) \tag{2.4}$$

3. CONTROLLABILITY AND OBSERVABILITY OF Δ-DIFFERENTIAL SYSTEMS

In this section, we prove necessary and sufficient conditions for controllability and observability of the system (1.1) and (1.2).

Definition 3.1. The Δ-differential systems S1 given by (1.1) and (1.2) is said to be completely controllable if for t_0, any initial state $(X(t_0)\otimes Y(t_0))=(P_0\otimes Q_0)$ and any given final state $(P_f\otimes Q_f)$ there exists a finite time $t_1 > t_0$ and a control $(U(t)\otimes V(t)), t_0\le t\le t_1$ such that $(X(t_1)\otimes Y(t_1))=(P_f\otimes Q_f)$.

Theorem 3.1. The time scale dynamical system S1 is completely controllable on the closed interval $J=[t_0, t_1]$ if and only if the $n^2\times n^2$ symmetric controllability matrix

$$M(t_0,t_1)=\int_{t_0}^{t_1}\phi(t_0,\sigma(s))(F\otimes G)(s)(F\otimes G)^*(s)\phi^*(t_0,\sigma(s))\Delta s] \tag{3.1}$$

where $\phi(t,\sigma(s))$ is defined in (2.3), is nonsingular. In this case the control

$$(U(t)\otimes V(t)) = -(F\otimes G)^{*}(t)\phi^{*}(t_0,\sigma(s))M^{-1}(t_0,t_1)$$
$$\{(P_0\otimes Q_0)-\phi(t_0,t_1)(P_f\otimes Q_f)\} \tag{3.2}$$

defined on $t_0 \le t \le t_1$, transfers $(X(t_0)\otimes Y(t_0)) = (P_0\otimes Q_0)$ to $(X(t_1)\otimes Y(t_1)) = (P_f\otimes Q_f)$.

Proof. Suppose that $M(t_0, t_1)$ is non singular, then the control defined by (3.2) exists. Now substituting (3.2) in (2.2) with $t = t_1$, we have

$$(X(t_1)\otimes Y(t_1)) = \phi(t_1,t_0)\left[P_0\otimes Q_0 - \int_{t_0}^{t_1} \begin{matrix}\phi(t_0,\sigma(s))(F\otimes G)(s)(F\otimes G)^{*}(s)\phi^{*} \\ \left\{(P_0\otimes Q_0)-\phi(t_0,t_1)(P_f\otimes Q_f)\right\}\Delta s\end{matrix}\right]$$
$$= \phi(t_1,t_0)\phi(t_0,t_1)$$
$$= (P_f\otimes Q_f)$$

hence the dynamical system S1 is completely controllable.

Next suppose that the dynamical system S1 is completely controllable on J, then we have to show that $M(t_0, t_1)$ is nonsingular. Then there exists a non zero $n^2\times 1$ vector α such that

$$\alpha^{*}M(t_0,t_1)\alpha = \int_{t_0} \alpha^{*}\phi(t_0,\sigma(s))(F\otimes G)(s)(F\otimes G)^{*}(s)\phi^{*}(t_0,\sigma(s))\alpha\Delta s$$
$$= \int_{t_0}^{t_1}\theta^{*}(\sigma(s),t_0)\theta(\sigma(s),t_0)\Delta s$$
$$= \int_{t_0}^{t_1}\|\theta\|^{2}\Delta s \ge 0 \tag{3.3}$$

where $\theta = (F\otimes G)^{*}(s)\phi^{*}(t_0,\sigma(s))\alpha$. From (3.3) $M(t_0, t_1)$ is positive semi definite.

Suppose that there exists some $\beta\neq 0$ such that $\beta^{*} M(t_0, t_1)\beta = 0$ then from (3.3) with $\theta = \eta$ when $\alpha = \beta$, implies

$$\int_{t_0}^{t_1}\|\eta\|^{2}\Delta s = 0$$

using the properties of norms, we have

$$\eta(\sigma(s),t_0) = 0, t_0 \le t \le t_1 \tag{3.4}$$

since S_1 is completely controllable, so there exists a control $(U(t)\otimes V(t))$ making $(X(t_1)\otimes Y(t_1)) = 0$ if $(X(t_0)\otimes Y(t_0)) = \beta$. Hence from (2.2) we have

$$\beta = -\int_{t_0}^{t_1} \phi(t_0,\sigma(s))(F \otimes G)(s)((U(s)\otimes V(s))\Delta s]$$

Consider

$$\|\beta\|^2 = \beta^*\beta = -\int_{t_0}^{t_1} ((U(s)\otimes V(s))^*(F\otimes G)^*(s)\phi^*(t_0,\sigma(s))\beta\Delta s$$

$$= -\int_{t_0}^{t_1} ((U(s)\otimes V(s))^*\eta(\sigma(s),t_0)\Delta s = 0$$

hence $\beta = 0$, which is a contradiction to our assumption. Thus M (t_0, t_1) is positive definite and is therefore non singular.

We now turn our attention to the concept of observability on a timescale dynamical system.

Definition 3.2. The timescale dynamical system (1.1) is completely observable on J = $[t_0; t_1]$ if for any time t_0 and any initial state $(X(t_0)\otimes Y(t_0) = (P_0 \otimes Q_0)$ there exists a finite time $t_1 > t_0$ such that the knowledge of $(U(t)\otimes V(t))$ and $(R(t)\otimes S(t))$ for $t_0 \le t \le t_1$ suffices to determine $(P_0 \otimes Q_0)$ uniquely.

Now we present a necessary and sufficient condition for the system (1.1) to be completely observable.

Theorem 3.2. The system S1 is completely observable on J if and only if the $n^2 \times n^2$ symmetric observability matrix

$$L(t_0,t_1) = \int_{t_0}^{t_1} \phi^*(s,t_0)(K\otimes L)^*(s)(K\otimes L)(s)\phi(s,t_0)\Delta s]$$

is non singular.

Proof. Suppose that $L(t_0, t_1)$ is non singular. It is simpler to consider the case of zero input, and it does not entail any loss of generality. Since the concept is not altered in the presence of a known input signal. Implies $(R(t)\otimes S(t)) = [K\otimes L](X(t)\otimes Y(t))$ since from $(X(t)\otimes Y(t)) = \phi(t,t_0)(P_0 \otimes Q_0)\psi^*(t,t_0)$ we have

$$(R(t)\otimes S(t)) = [K\otimes L]\phi(t,t_0)(P_0 \otimes Q_0)\psi^*(t,t_0) \tag{3.5}$$

pre multiplying (3.5) with $\phi^*(t,t_0)(K\otimes L)^*(t)$, post multiply with $\psi^*(t_0,t)$ and integrating from t_0 to t_1 we obtain

$$\int_{t_0}^{t_1} \phi^*(s,t_0)(K \otimes L)^*(s)(R(s) \otimes S(s))\Delta s = \mathrm{L}(\mathrm{t}_0,t_1)(P_0 \otimes \mathrm{Q}_0)$$

since L(t0, t1) is non singular, $(P_0 \otimes \mathrm{Q}_0)$ can be determined uniquely. Hence the dynamical system S1 is completely observable.

Conversely suppose that the dynamical system S1 is completely observable. Then we prove that $\mathrm{L}(\mathrm{t}_0, \mathrm{t}_1)$ is non singular. Since $\mathrm{L}(\mathrm{t}_0, \mathrm{t}_1)$ is symmetric, we can construct the quadratic form

$$\alpha^* L(t_0,t_1)\alpha = \int_{t_0}^{t_1} \alpha^* \phi^*(s,t_0)(F \otimes G)^*(s)(F \otimes G)(s)\phi(t_0,s)\alpha\Delta s$$

$$= \int_{t_0}^{t_1} \left\|\eta(s,t_0)\right\|^2 \Delta s \geq 0 \tag{3.6}$$

where α is an arbitrary column n^2-vector and $\eta(s,t_0) = (K \otimes L)(s)\phi(t_0,s)\alpha$. From (3.6) $\mathrm{L}(\mathrm{t}_0, \mathrm{t}_1)$ is positive semi definite. Suppose that there exists some $\beta \neq 0$ such that $\beta^* \mathrm{L}(\mathrm{t}_0, \mathrm{t}_1)\beta = 0$ then from (3.6) with $\eta=\theta$ when $\alpha=\beta$, implies

$$= \int_{t_0}^{t_1} \left\|\theta(s,t_0)\right\|^2 \Delta s = 0 \Rightarrow \theta(\mathrm{s},\mathrm{t}_0) = 0, \mathrm{t}_0 \leq \mathrm{s} \leq \mathrm{t}_1.$$

$$\Rightarrow (K \otimes L)(s)\varphi(t_0,s)\beta = 0, \mathrm{t}_0 \leq \mathrm{s} \leq \mathrm{t}_1$$

From (3.5), this implies that when $(P_0 \otimes \mathrm{Q}_0) = \beta$, the out put is identically zero throughout the interval, so that $(P_0 \otimes \mathrm{Q}_0)$ can not be determined in this case from knowledge of $(R(t) \otimes S(t))$.

This contradicts the supposition that S1 is completely observable.

Hence $\mathrm{L}(\mathrm{t}_0, \mathrm{t}_1)$ positive definite, therefore non-singular. The proof is complete.

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