

# On Inequalities Involving Moments of Discrete Uniform Distributions

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## Abstract

Some inequalities for the moments of discrete uniform distributions are obtained. The inequalities for the ratio and difference of moments are given. The special cases give the inequalities for the standard power means.

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## 1. INTRODUCTION

Let  $x_1, x_2, \dots, x_n$  denote  $n$  real numbers such that  $a \leq x_i \leq b, i = 1, 2, \dots, n$ . The  $r^{\text{th}}$  order moment  $\mu_r'$  of these numbers is defined as

$$\mu_r' = \frac{1}{n} \sum_{i=1}^n x_i^r. \quad (1.1)$$

The power mean of order  $r$  namely  $M_r$  is defined as

$$M_r = (\mu_r')^{1/r} \quad \text{for } r \neq 0 \quad (1.2)$$

and

$$M_r = \lim_{r \rightarrow 0} (\mu_r')^{1/r} \quad \text{for } r = 0. \quad (1.3)$$

It may be noted that  $M_{-1}, M_0$  and  $M_1$  respectively define harmonic mean, geometric mean and Arithmetic mean. It is well known that the power mean  $M_r$  is an increasing function of  $r$ .

For  $0 < a \leq x_i \leq b, i = 1, 2, \dots, n$ , we have, [1],

$$\mu_r' \geq (\mu_s')^{r/s}, \quad (1.4)$$

where  $r$  is a positive real number and  $s$  is any real number such that  $r > s$ . If  $r$  is negative real number with  $r > s$ , the reverse inequality holds. For  $s = 0$ , the inequality (1.4) gives

$$\mu'_r \geq (M_o)^r \tag{1.5}$$

See also [2]. Some bounds for the difference and ratio of moments have also been investigated in literature; see [3-6].

Our main results give the refinements of the inequalities (1.4) and (1.5) when the minimum and maximum values of  $x_1$  namely,  $a$  and  $b$ , and the value of  $n$  is prescribed, (Theorem 2.1 and 2.2, below). The bounds for the difference and ratio of moments are obtained (Theorem 2.3-2.6, below). We also discuss the cases when the inequalities reduce to equalities. As the special cases, we get various bounds connecting lower order moments and the standard means of the  $n$  real numbers, (Inequalities 3.1 - 3.33, below), also see [7-9].

## 2. MAIN RESULTS

**Theorem 2.1.** For  $0 < a \leq x_i \leq b, i = 1, 2, \dots, n$ , we have

$$\mu'_r \geq \frac{a^r + b^r}{n} + \left( \frac{n}{n-2} \right)^{\frac{r-s}{s}} \left[ \mu'_s - \frac{a^s + b^s}{n} \right]^{\frac{r}{s}}, \tag{2.1}$$

where  $n \geq 3, r$  is a positive real number and is any non-zero real number such that  $r > s$  For  $s < r < 0$  the reverse inequality holds. The inequality (2.1) becomes equality when

$$x_1 = a, x_2 = x_3 = \dots = x_{n-1} = \left[ \frac{n\mu'_s - a^s - b^s}{n-2} \right]^{\frac{1}{s}} \text{ and } x_n = b.$$

The inequality (2.1) provides a refinement of the inequality (1.4).

**Proof.** The  $r^{\text{th}}$  order moment of  $n$  real numbers  $x_i$ , with  $x_1 = a$  and  $x_n = b$  can be written as

$$\mu'_r = \frac{a^r + b^r}{n} + \left( \frac{n-2}{n} \right) \left[ \frac{x_2^r + \dots + x_{n-1}^r}{n-2} \right]. \tag{2.2}$$

Apply (1.4) to  $n - 2$  real numbers  $x_2, x_3, \dots, x_{n-1}$ , we find that

$$\frac{x_2^r + \dots + x_{n-1}^r}{n-2} \geq \left[ \frac{x_2^s + \dots + x_{n-1}^s}{n-2} \right]^{\frac{r}{s}}, \tag{2.3}$$

where  $r$  is a positive real number and  $s$  is any non zero real number such that  $r > s$ .

Combine (2.2) and (2.3), we get

$$\mu'_r \geq \frac{a^r + b^r}{n} + \left( \frac{n-2}{n} \right) \left[ \frac{x_2^s + \dots + x_{n-1}^s}{n-2} \right]^{\frac{r}{s}}. \tag{2.4}$$

Also,

$$\mu'_s = \frac{1}{n} [a^s + x_2^s + \dots + b^s], \quad (2.5)$$

Therefore

$$x_2^s + x_3^s + \dots + x_{n-1}^s = n\mu'_s - a^s - b^s \quad (2.6)$$

Substituting the value from (2.6) in (2.4), we immediately get (2.1).

For  $s < r < 0$ , inequality (2.3) reverses its order [2]. It follows therefore that inequality (2.1) will also reverse its order for  $s < r < 0$ .

**Theorem 2.2.** For  $0 < a \leq x_i \leq b, i = 1, 2, \dots, n$ , we have

$$\mu'_r \geq \frac{a^r + b^r}{n} + \frac{n-2}{n} \left( \frac{M_0^n}{ab} \right)^{\frac{r}{n-2}}, \quad (2.7)$$

where  $n \geq 3$  and  $r$  is a non-zero real number.

The inequality (2.7) becomes equality when

$$x_1 = a, x_2 = x_3 = \dots = x_{n-1} = \left[ \frac{M_0^n}{ab} \right]^{\frac{1}{n-2}} \text{ and } x_n = b.$$

The inequality (2.1) provides a refinement of the inequality (1.5).

**Proof.** Apply (1.5) to  $n - 2$  real numbers  $x_2, x_3, \dots, x_{n-1}$ , we find that

$$\frac{x_2^r + x_3^r + \dots + x_{n-1}^r}{n-2} \geq \left[ \frac{1}{x_2^{n-2}} \cdot \frac{1}{x_3^{n-2}} \dots \frac{1}{x_{n-1}^{n-2}} \right]^{\frac{r}{n-2}}. \quad (2.8)$$

Combine (2.2) and (2.8), we get

$$\mu'_r \geq \frac{a^r + b^r}{n} + \frac{n-2}{n} \left[ \frac{1}{x_2^{n-2}} \cdot \frac{1}{x_3^{n-2}} \dots \frac{1}{x_{n-1}^{n-2}} \right]^{\frac{r}{n-2}}. \quad (2.9)$$

Also,

$$M_0 = (a \cdot x_2 \cdot \dots \cdot b)^{\frac{1}{n}}, \quad (2.10)$$

therefore

$$x_2 x_3 \dots x_{n-1} = \frac{M_0^n}{ab}. \quad (2.11)$$

Substituting the value from (2.11) in (2.9) we immediately get (2.7).

**Theorem 2.3.** For  $0 < a \leq x_i \leq b, i = 1, 2, \dots, n$ , we have

$$\mu'_r - (\mu'_s)^{\frac{r}{s}} \geq \frac{2}{n} \left[ \frac{a^r + b^r}{2} - \left( \frac{a^s + b^s}{2} \right)^{\frac{r}{s}} \right]. \quad (2.12)$$

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where  $r$  is a positive real number and  $s$  is a nonzero real number such that  $r > s$ . For  $s < r < 0$ , the reverse inequality holds.

For  $n = 2$  inequality (2.13) becomes equality. For  $n \geq 3$ , inequality (2.13) is sharp; equality holds when

$$x_1 = a, x_2 = x_3 = \dots = x_{n-1} = \left( \frac{a^s + b^s}{2} \right)^{\frac{1}{s}} \text{ and } x_n = b.$$

**Proof.** It follows from the inequality (2.1) that

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$$\mu'_r - (\mu'_s)^{\frac{r}{s}} \geq \frac{a^r + b^r}{n} + \left( \frac{n}{n-2} \right)^{\frac{r-s}{s}} \left[ \mu'_s - \frac{a^s + b^s}{n} \right]^{\frac{r}{s}} - (\mu'_s)^{\frac{r}{s}}.$$

Consider a function

$$f(\mu'_s) = \frac{a^r + b^r}{n} + \left( \frac{n}{n-2} \right)^{\frac{r-s}{s}} \left[ \mu'_s - \frac{a^s + b^s}{n} \right]^{\frac{r}{s}} - (\mu'_s)^{\frac{r}{s}}. \quad (2.13)$$

The function  $f(\mu'_s)$  is continuous in the interval  $[a^s, b^s]$  and its derivative

$$\frac{df}{d\mu'_s} = \frac{r}{s} \left( \frac{n}{n-2} \right)^{\frac{r-s}{s}} \left[ \mu'_s - \frac{a^s + b^s}{n} \right]^{\frac{r-s}{s}} - \frac{r}{s} (\mu'_s)^{\frac{r-s}{s}}. \quad (2.14)$$

vanishes at

$$\mu'_s = \frac{a^s + b^s}{2} \quad (2.15)$$

The value of the second order derivative

$$\frac{d^2f}{d\mu_s'^2} = \frac{r}{s} \frac{r-s}{s} \left[ \left( \frac{n}{n-2} \right)^{\frac{r-s}{s}} \left( \mu'_s - \frac{a^s + b^s}{n} \right)^{\frac{r-2s}{s}} - (\mu'_s)^{\frac{r-2s}{s}} \right], \quad (2.16)$$

at  $\mu'_s = \frac{a^s + b^s}{2}$  is

$$\frac{d^2f}{d\mu_s'^2} = \frac{r(r-s)}{s^2} \frac{2}{n-2} \left( \frac{a^s + b^s}{2} \right)^{\frac{r-2s}{s}} \quad (2.17)$$

It follows from (2.17) that

$$\frac{d^2f}{d\mu_s'^2} > 0 \quad \text{for } r > 0$$

and

$$\frac{d^2f}{d\mu_s'^2} < 0 \text{ for } r < 0.$$

Therefore, for  $r > 0$ , the function  $f(\mu_s')$  achieves its minimum when the value of  $\mu_s'$  is given by (2.15). Hence

$$f(\mu_s') \geq \frac{2}{n} \left[ \frac{a^r + b^r}{2} \right] - \left( \frac{a^s + b^s}{2} \right)^{\frac{r}{s}}.$$

This proves (2.12).

Likewise, for  $r < 0$ , the function  $f(\mu_s')$  achieves its maximum when the value of  $\mu_s'$  is given by (2.15) and we conclude that

$$\mu_r' - (\mu_s')^{\frac{r}{s}} \leq \frac{2}{n} \left[ \frac{a^r + b^r}{2} - \left( \frac{a^s + b^s}{2} \right)^{\frac{r}{s}} \right].$$

**Theorem 2.4.** For  $0 < a \leq x_i \leq b$ ,  $i = 1, 2, \dots, n$ , we have

$$\mu_r' - M_0^r \geq \frac{2}{n} \left[ \frac{a^r + b^r}{2} - (\sqrt{ab})^r \right] \tag{2.18}$$

where  $r$  is a non zero real number.

For  $n = 2$ , the inequality (2.19) becomes equality. For  $n \geq 3$ , inequality (2.19) is sharp, equality holds when

$$x_1 = a, x_2 = x_3 = \dots = x_{n-1} = \sqrt{ab} \text{ and } x_n = b.$$

**Proof.** It follows from the inequality (2.7) that

$$\mu_r' - M_0^r \geq \frac{a^r + b^r}{n} + \frac{n-2}{n} \left( \frac{M_0^n}{ab} \right)^{\frac{r}{n-2}} - M_0^r.$$

Consider a function

$$g(M_0) = \frac{a^r + b^r}{n} + \frac{n-2}{n} \left( \frac{M_0^n}{ab} \right)^{\frac{r}{n-2}} - M_0^r. \tag{2.19}$$

The derivative

$$\frac{dg}{dM_0} = rM_0^{r-1} \left[ \left( \frac{M_0}{\sqrt{ab}} \right)^{\frac{2r}{n-2}} - 1 \right] \tag{2.20}$$

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vanishes at

$$M_0 = \sqrt{ab} \quad (2.21)$$

The value of the second derivative

$$\frac{d^2 g}{dM_0} = rM_0^{r-2} \left[ \frac{n(r-1+2)}{n-2} \left( \frac{M_0}{\sqrt{ab}} \right)^{\frac{2r}{n-2}} - (r-1) \right]. \quad (2.22)$$

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at  $M_0 = \sqrt{ab}$  is

$$\frac{d^2 g}{dM_0^2} = \frac{2r^2}{n-2} (ab)^{\frac{r-2}{2}}.$$

Clearly,

$$\frac{d^2 g}{dM_0^2} > 0 \quad \text{for } M_0 = \sqrt{ab}.$$

Therefore, for  $r \neq 0$ , the function  $g(M_0)$  attains its minimum at  $M_0 = \sqrt{ab}$  and we have

$$g(\mu'_s) \geq \frac{2}{n} \left[ \frac{a^r + b^r}{2} - (\sqrt{ab})^r \right].$$

This proves (2.18).

**Theorem 2.5.** For  $0 < a \leq x_i \leq b$  for  $i = 1, 2, \dots, n$ , we have

$$\frac{\mu'_r}{(\mu'_s)^{r/s}} \geq \frac{a^r + b^r}{a^s + b^s} \left\{ \frac{1}{n} \left[ a^s + b^s + (n-2) \left( \frac{a^r + b^r}{a^s + b^s} \right)^{\frac{s}{r-s}} \right] \right\}^{\frac{s-r}{s}}. \quad (2.23)$$

where  $r$  is a positive real number and  $s$  is any non-zero real number such that  $r > s$ .

For  $s < r < 0$ , the reverse inequality holds.

For  $n = 2$ , the inequality (2.23) becomes equality. For  $n \geq 3$ , this inequality is sharp; equality

$$\text{holds when } x_1 = a, x_2 = x_3 = \dots = x_{n-1} = \left( \frac{a^r + b^r}{a^s + b^s} \right)^{\frac{1}{r-s}} \text{ and } x_n = b.$$

**Proof.** It follows from inequality (2.1) that

$$\frac{\mu'_r}{(\mu'_s)^{r/s}} \geq \left[ \frac{a^r + b^r}{n} + \left( \frac{n}{n-2} \right)^{\frac{r-s}{s}} \left( \mu'_s - \frac{a^s + b^s}{n} \right)^{r/s} \right] \frac{1}{(\mu'_s)^{r/s}}, \quad (2.24)$$

where  $r$  is a positive real number and  $s$  is a non zero real number such that  $r > s$ . We now find the minimum value of the right side expression in (2.24) as  $\mu'_s$  varies over the interval  $[a^s, b^s]$ .

Consider a function

$$h(\mu'_s) = \left[ \frac{a^r + b^r}{n} + \left( \frac{n}{n-2} \right)^{\frac{r-s}{s}} \left( \mu'_s - \frac{a^s + b^s}{n} \right)^{r/s} \right] \frac{1}{(\mu'_s)^{r/s}}. \quad (2.25)$$

The function  $h(\mu'_s)$  is continuous in the interval  $[a^s, b^s]$  and its derivative

$$\frac{dh}{d\mu'_s} = \frac{r(a^s + b^s)}{sn} \left( \frac{1}{\mu'_s} \right)^{\frac{r+s}{s}} \left[ \left( \frac{n\mu'_s - a^s - b^s}{n-2} \right)^{\frac{r-s}{s}} - \frac{a^r + b^r}{a^s + b^s} \right] \quad (2.26)$$

vanishes at

$$\mu'_s = \frac{a^s + b^s}{n} + \frac{n-2}{n} \left( \frac{a^r + b^r}{a^s + b^s} \right)^{\frac{s}{r-s}}. \quad (2.27)$$

From (2.26), if  $r$  is a positive real number and  $s$  is any non zero real numbersuch that  $r > s$  then the sign of  $\frac{dh}{d\mu'_s}$  changes from negative to positive while  $\mu'_s$  passes through the value given by (2.27). The function  $h(\mu'_s)$  achieves its minimum at the value of  $\mu'_s$  given by (2.27). We therefore have,

$$h(\mu'_s) \geq \frac{a^r + b^r}{a^s + b^s} \left\{ \frac{1}{n} \left[ a^s + b^s + (n-2) \left( \frac{a^r + b^r}{a^s + b^s} \right)^{\frac{s}{r-s}} \right] \right\}^{\frac{s-r}{s}}. \quad (2.28)$$

Combining (2.24), (2.25) and (2.28), the inequality (2.24) follows immediately.

For  $s < r < 0$ , it follows from (2.26) that the sign of  $\frac{dh}{d\mu'_s}$  changes from positive to negative while  $\mu'_s$  passes through the value given by (2.27). In this case function  $f(\mu'_s)$  attains its maximum at the value of  $\mu'_s$  given by (2.27). We therefore have,

$$h(\mu'_s) \leq \frac{a^r + b^r}{a^s + b^s} \left\{ \frac{1}{n} \left[ a^s + b^s + (n-2) \left( \frac{a^r + b^r}{a^s + b^s} \right)^{\frac{s}{r-s}} \right]^{\frac{s-r}{s}} \right\}. \quad (2.29)$$

On the other hand we conclude from theorem 2.1 that

$$\frac{\mu'_r}{(\mu'_s)^{\frac{r}{s}}} \leq \left[ \frac{a^r + b^r}{n} + \left( \frac{n}{n-2} \right)^{\frac{r-s}{s}} \left( \mu'_s - \frac{a^s + b^s}{n} \right)^{\frac{r}{s}} \right] \frac{1}{(\mu'_s)^{\frac{r}{s}}}. \quad (2.30)$$

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Combining (2.25), (2.26) and (2.27), we find that for  $s < r < 0$ ,

$$\frac{\mu'_r}{(\mu'_s)^{\frac{r}{s}}} \leq \left[ \frac{a^r + b^r}{n} + \left( \frac{n}{n-2} \right)^{\frac{r-s}{s}} \left( \mu'_s - \frac{a^s + b^s}{n} \right)^{\frac{r}{s}} \right] \frac{1}{(\mu'_s)^{\frac{r}{s}}}.$$

**Theorem 2.6** For  $0 < a \leq x_i \leq b$  for  $i = 1, 2, \dots, n$ , we have

$$\frac{\mu'_r}{M_0^r} \geq \left[ \frac{a^r + b^r}{2(\sqrt{ab})^r} \right]^{\frac{2}{n}} \quad (2.31)$$

where  $r$  is a non zero real number.

For  $n = 2$ , the inequality (2.31) becomes equality. For  $n \geq 3$ , the inequality (2.31) is sharp; equality holds when

$$x_1 = a, x_2 = x_3 = \dots = x_{n-1} = \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} \text{ and } x_n = b.$$

**Proof.** It follows from inequality (2.7) that

$$F(M_0) = \left[ \frac{a^r + b^r}{2} + \frac{n-2}{n} \left( \frac{M_0^n}{ab} \right)^{\frac{r}{n-2}} \right] \frac{1}{M_0^r}. \quad (2.32)$$

where  $r$  is a non zero real number. We now find the minimum value of the right hand side expression in (2.33).

Consider a function

$$\frac{\mu'_r}{M_0^r} \geq \left[ \frac{a^r + b^r}{n} + \frac{n-2}{n} \left( \frac{M_0^n}{ab} \right)^{\frac{r}{n-2}} \right] \frac{1}{M_0^r} \quad (2.33)$$

The derivative

$$\frac{dF}{dM_0} = \frac{2r}{nM_0^{r+1}} \left[ \left( \frac{M_0^n}{ab} \right)^{\frac{r}{n-2}} - \frac{a^r + b^r}{2} \right]. \quad (2.34)$$



vanishes at

$$M_0 = (ab)^{\frac{1}{n}} \left( \frac{a^r + b^r}{2} \right)^{\frac{n-2}{nr}}. \quad (2.35)$$

From (2.34), if  $r$  is a non zero real number then the sign of  $\frac{dF}{dM_0}$  changes from negative to positive while  $M_0$  passes through the value given by (2.35). The function  $f(M_0)$  attains its minimum at the value of  $M_0$  given by (2.35). We therefore have

$$F(M_0) \geq \left[ \frac{a^r + b^r}{2(\sqrt{ab})^r} \right]^{\frac{2}{n}}. \quad (2.36)$$

Combining (2.32), (2.33) and (2.36), inequality (2.31) follows immediately.

### 3. SOME SPECIAL CASES

From the application point of view, it is of interest to know the bounds for the first four moments and the bounds for standard power means (namely, Arithmetic mean, ( $\mu'_1 = \bar{x}$ ), Geometric mean (G) and Harmonic mean (H)). These bounds are also of fundamental interest in the theory of inequalities. By assigning particular values to  $r$  and  $s$ , in the generalized inequalities obtained in this paper, we can find inequalities connecting various power means and moments. The following inequalities can be deduced easily from the generalized inequalities given in Theorems 2.1 and 2.2:

$$\mu'_2 \geq \frac{a^2 + b^2}{n} + \frac{n}{n-2} \left[ \mu'_1 - \frac{a+b}{n} \right]^2, \quad (3.1)$$

$$\mu'_2 \geq \frac{a^2 + b^2}{n} + \left( \frac{n-2}{n} \right)^3 \left[ \frac{abnH}{abn - (a+b)H} \right]^2, \quad (3.2)$$

$$\mu'_2 \geq \frac{a^2 + b^2}{n} + \frac{n-2}{n} \left[ \frac{G^n}{ab} \right]^{\frac{2}{n-2}}, \quad (3.3)$$

$$\mu'_1 \geq \frac{a+b}{n} + \left( \frac{n-2}{n} \right)^2 \left[ \frac{abnH}{abn - (a+b)H} \right], \quad (3.4)$$

$$\mu'_1 \geq \frac{a+b}{n} + \frac{n-2}{n} \left[ \frac{G^n}{ab} \right]^{\frac{1}{n-2}}, \quad (3.5)$$

$$G \geq (ab)^{\frac{1}{n}} \left[ \frac{n-2}{\frac{n}{H} - \frac{a+b}{ab}} \right]^{\frac{n-2}{n}}, \quad (3.6)$$

$$\mu'_3 \geq \frac{a^3 + b^3}{n} + \left( \frac{n}{n-2} \right)^2 \left[ \mu'_1 - \frac{a+b}{n} \right]^3, \quad (3.7)$$

$$\mu'_3 \geq \frac{a^3 + b^3}{n} + \sqrt{\frac{n}{n-2}} \left[ \mu'_2 - \frac{a^2 + b^2}{n} \right]^{\frac{3}{2}}, \quad (3.8)$$

$$\mu'_4 \geq \frac{a^4 + b^4}{n} + \left( \frac{n}{n-2} \right)^3 \left[ \mu'_1 - \frac{a+b}{n} \right]^4, \quad (3.9)$$

$$\mu'_4 \geq \frac{a^4 + b^4}{n} + \frac{n}{n-2} \left[ \mu'_2 - \frac{a^2 + b^2}{n} \right]^2 \quad (3.10)$$

and

$$\mu'_4 \geq \frac{a^4 + b^4}{n} + \sqrt[3]{\frac{n}{n-2}} \left[ \mu'_3 - \frac{a^3 + b^3}{n} \right]^{\frac{4}{3}}. \quad (3.11)$$

The following lower bounds for the difference of moments and means are deduced from the generalized inequalities given in Theorems 2.3 and 2.4:

$$\mu'_2 - \mu_1'^2 \geq \frac{1}{2n} (b-a)^2, \quad (3.12)$$

$$\mu'_2 - H^2 \geq \frac{(b-a)^2}{n} \left[ 1 + \frac{2ab}{(a+b)^2} \right], \quad (3.13)$$

$$\mu'_2 - G^2 \geq \frac{(b-a)^2}{n}, \quad (3.14)$$

$$\mu'_1 - H \geq \frac{(b-a)^2}{n(a+b)}, \quad (3.15)$$

$$\mu'_1 - G \geq \frac{(\sqrt{b} - \sqrt{a})^2}{n}, \quad (3.16)$$

$$G - H \geq \left( \frac{2ab}{a+b} \right)^{\frac{n-2}{n}} \left[ (\sqrt{ab})^{\frac{2}{n}} - \left( \frac{2ab}{a+b} \right)^{\frac{1}{n}} \right], \quad (3.17)$$

$$\mu'_3 - \mu_1'^3 \geq \frac{3}{4} \frac{a+b}{n} (b-a)^2, \quad (3.18)$$

$$\mu_3' - \mu_2'^{3/2} \geq \frac{2}{n} \left[ \frac{a^3 + b^3}{2} - \left( \frac{a^2 + b^2}{2} \right)^{3/2} \right], \quad (3.19)$$

$$\mu_4' - \mu_1'^4 \geq \frac{2}{n} \left[ \frac{a^4 + b^4}{2} - \left( \frac{a + b}{2} \right)^4 \right], \quad (3.20)$$

$$\mu_4' - \mu_2'^2 \geq \frac{2}{n} \left[ \frac{a^4 + b^4}{2} - \left( \frac{a^2 + b^2}{2} \right)^2 \right] \quad (3.21)$$

and

$$\mu_4' - \mu_3'^{4/3} \geq \frac{2}{n} \left[ \frac{a^4 + b^4}{2} - \left( \frac{a^3 + b^3}{2} \right)^{4/3} \right]. \quad (3.22)$$

The following lower bounds for the ratio of moments and means are deduced from the generalized inequalities given in Theorems 2.5 and 2.6:

$$\frac{\mu_2'}{\mu_1'^2} \geq \frac{(a^2 + b^2)n}{(a + b)^2 + (n - 2)(a^2 + b^2)}, \quad (3.23)$$

$$\frac{\mu_3'}{\mu_1'^3} \geq \frac{(a^2 - ab + b^2)n^2}{\left[ a + b + (n - 2)\sqrt{a^2 - ab + b^2} \right]^2}, \quad (3.24)$$

$$\frac{\mu_3'}{\mu_2'^{3/2}} \geq \frac{\sqrt{n}(a^3 + b^3)}{\left[ (a^2 + b^2)^3 + (n - 2)(a^3 + b^3)^2 \right]^{1/2}}, \quad (3.25)$$

$$\frac{\mu_4'}{\mu_1'^4} \geq \frac{n^3(a^4 + b^4)}{\left[ (a + b)^{4/3} + (n - 2)(a^4 + b^4)^{1/3} \right]^3}, \quad (3.26)$$

$$\frac{\mu_4'}{\mu_2'^2} \geq \frac{n(a^4 + b^4)}{(a^2 + b^2)^2 + (n - 2)(a^4 + b^4)}, \quad (3.27)$$

$$\frac{\mu_4'}{\mu_3'^{4/3}} \geq \frac{\sqrt[3]{n}(a^4 + b^4)}{\left[ (a^3 + b^3)^4 + (n - 2)(a^4 + b^4)^3 \right]^{1/3}}, \quad (3.28)$$

$$\frac{\mu_1'}{H} \geq \frac{1}{n^2} \left( n - 2 + \frac{a + b}{\sqrt{ab}} \right)^2, \quad (3.29)$$

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$$\frac{\mu'_1}{G} \geq \left[ \frac{a+b}{2\sqrt{ab}} \right]^{\frac{2}{n}}, \quad (3.30)$$

$$\frac{G}{H} \geq \left[ \frac{a+b}{2\sqrt{ab}} \right]^{\frac{2}{n}}, \quad (3.31)$$

$$\frac{\mu'_2}{G^2} \geq \left[ \frac{a^2+b^2}{2ab} \right]^{\frac{2}{n}} \quad (3.32)$$

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and

$$\frac{\mu'_2}{H^2} \geq \frac{1}{n^3} \left[ \left( \frac{(a^2+b^2)(a+b)^2}{a^2b^2} \right)^{\frac{1}{3}} + n - 2 \right]^3. \quad (3.33)$$

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