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A Derivative Free Hybrid Equation Solver by Alloying of the Conventional Methods

Amit Kumar Maheshwari
Advanced Materials and Processes Research Institute (CSIR),
Bhopal, India 462064
Tel.: +919827312078
Email: ampri.amit@gmail.com

Abstract

This paper pronounces a modified numerical scheme to the conventional formula of Newton-Raphson for solving the nonlinear and transcendental equations especially those which cannot be solved by the basic algebra. Finding the derivative of a function is difficult in some case of problems. The present formula is made with the target to aloof the need of obtaining the derivative of the function. Comparative analysis shows that the present method is faster than Newton-Raphson method, Adomian method, Rabolian method, Abbasbandy method, Basto method & Feng method. Iteration cost effective parameters – number of iteration steps & the value of effective error is also found to be minimum than these methods.

Keywords: Algebraic & Transcendental equations, Bisection method, Regula-Falsi method, Newton-Raphson method, Iteration Process, Derivative free methods.

1. Introduction

The laws of basic mathematics are unable to solve many equations of the form $f(x) = 0$ which contain the partial or full involvement of the terms containing non-linear algebraic functions particularly of higher order and having transcendental terms. Numerical analysis is a branch of mathematics which is generally used to solve those algebraic and transcendental equations, which are difficult to solve by usual Mathematical methods. Methods like Bisection, Regula falsi and Newton-Raphson are oftenly used for this purpose. The various methods are discussed below.

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1.1 Bisection Method

This method is based on finding the particular root between two easy roots confining it. Here, the actual root bracketed between two points say x_i and x_{i+1} give the mid-point $x_{m(i,i+1)} = \dfrac{x_i + x_{i+1}}{2}$ between them and lead to generate two new intervals,

1. x_i and $x_{m(i,i+1)}$
2. x_{i+1} and $x_{m(i,i+1)}$

Thus, there can be two cases

$$f(x_i)f(x_{m(i,i+1)}) = \begin{cases} <0 & i.e.\ the\ product\ is\ negative \\ >0 & i.e.\ the\ product\ is\ positive \quad For\ i = 1\ or\ 2 \end{cases} \tag{1.1}$$

In the first case we say the actual root lies between x_i and $x_{m(i,i+1)}$ where as the second case escapes the root to be within x_i and $x_{m(i,i+1)}$ and fall it in the next interval. This strategy can be used to further refine the result to its actual convergence at last.

1.2 Regula-Falsi Method

This method is based on the strategy of assumption in bisection method and hence considered as the improvement of above method. Here, the process algorithm involve the following modified formula for $x_{m(\,i,\,i\,,+1)}$

$$x_{m(i,i+1)} = x_i - \frac{f(x_i)(x_{i+1} - x_i)}{f(x_{i+1}) - f(x_i)} \tag{1.2}$$

Bracketing of roots is made exactly in the same way as bisection method.

1.3 Newton-Raphson Method

Both the above methods require the consideration of two initial points in each proceeding step. This method is advancement to both the methods and requires the input of a single point. Besides, here the result obtained in one step serve the input point for the next step and there is no need to check the property of obtained root each time. The formula defining its equation is as follows,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{1.3}$$

This method requires the computation of one derivative but its convergence is far better than Bisection and Regula falsi method. For this reason, Newton-Raphson is considered as the best among the three methods.

Many times it is quite difficult to obtain the derivative of a function due to its complicated nature. In those cases, the situation deprives to use the Newton-Raphson method which is considered to be best among the conventional methods. The present formula is made with the target to obtain the result of the targeted problems in all such situations. Comparative analysis shows that the present method is faster than many methods of its class i.e. Newton-Raphson method, Adomian method, Rabolian method, Abbasbandy method, Basto method & Feng method. Iteration cost effective parameters – number of iteration steps & the value of effective error is also found to be minimum than these methods.

2. Formula Derivation

Let, us assuming the function defining the equation $f(X) = 0$ is continuous and differentiable in the whole domain and at least in the part of domain in which the real root of the equation lies. The present iteration result runs through the basic strategies of Newton-Raphson method. So, assuming two points $(X_{n0}, f(X_{n0}))$ and $(X_{n1}, f(X_{n1}))$ such that $f(X_{n0})$ and $f(X_{n1})$ are opposite in sign and the ordinate part of the given equation $f(X) = 0$ follows the inequality $|f(X_{n0})| < |f(X_{n1})|$. In this way, from the Taylor's expansion of $f(X)$ about the point "X_{n0}" we have

$$f(x) = f(x_{n0}) + \frac{X - X_{n0}}{1!} f'(X_{n0}) + \frac{(X - X_{n0})^2}{2!} f^n(x_{n0}) + \dots\dots \quad (2.1)$$

Where, X_{n0} is the initial approximation to the root of the equation $f(X) = 0$.

Let's approximate $f'(X_{n0})$ as $\dfrac{f(X_n)_{Regula\text{-}falsi} - f(X_n)_{Bisection}}{(X_n)_{Regula\text{-}falsi} - (X_n)_{Bisection}}$ and say it $f(z)$ for convenience. Further, if X_n constitutes a root of the equation $f(X) = 0$ in the open interval I, in which the function is continuous and has well defined first order derivative. Then,

$$f(X_n) = f(X_{n0}) + \frac{(X_n - X_{n0})}{1!} f(z) + \frac{(X_n - X_{n0})^2}{2!} D(f(z)) + \dots\dots \quad (2.2)$$

Using the fact $f(X_n) = 0$ equation (2.2) becomes

$$0 = f(X_{n0}) + \frac{(X_n - X_{n0})}{1!} f(z) + \frac{(X_n - X_{n0})^2}{2!} D(f(z)) + \dots\dots \quad (2.3)$$

Considering only the linear terms of equation (2.3), the value of the root X_n can be obtained,

$$X_n = \frac{X_{n0} f(z) - f(X_{n0})}{f(z)} \quad (2.4)$$

3. Process Algorithm

i) Choose any two points say X_0 and X_1 such that $f(X_0)$ and $f(X_1)$ are opposite in sign. The ordinate with lesser $f(X)$ is assigned X_{n0} and the second to be X_{n1}.

ii) Find the results by Bisection method (equation (1.1)), Regula Falsi method (equation (1.2)) and present formula (equation (2.4)) for the two points X_{n0} and X_{n1}.

iii) The Following iterations were made by taking X_{n0} as the result of present formula, X_{n1} as the Result of Regula Falsi formula through redefining *f(z)* as $= \dfrac{f(X_{n0}) - f(X_{n1})}{(X_{n0}) - (X_{n1})}$.

iv) The process is repeated to get the result from next iteration step which in turn will give more exact result.

v) The process terminates when required level of accuracy is obtained.

4. Numerical Results and Discussions

Following illustrative example defines the result obtained by present method to solve targeting type of equations. At the same time, the results are compared with some other recent methods to compare the efficiency and accuracy of it.

Illustrative problem 1: Consider the following equation

$$f(x) = x^2 - (1-x)^5 = 0 \tag{8}$$

Taking $x_0 = 0.2$ as the initial approximation. The results obtained by Newton-Raphson method, Adomian method, Rabolian method, Abbasbandy method, Basto method, Feng method and the present method is shown in Table 1

Table-1: Comparison of the results of $f(x) = x^2 - (1-x)^5 = 0$ with different methods

Method	**Number of Iterations**	X_n	$f(X_n)$
Newton-Raphson	03	0.345953774	−1.7E-06
Adomian	10	0.340622225	−0.00862
Rabolian	05	0.346021366	0.000107
Abbasbandy	02	0.345954646	−2.7E-07
Basto	02	0.345952219	−4.2E-06
Feng	03	0.345954816	−7.8E-11
Present	04	0.345954816	0

Illustrative problem 2: Consider again the equation

$$f(x) = e^{x} - 3x^{2} = 0 \tag{9}$$

Taking $x_0 = 0$ as the initial approximation. Various results of different methods and the present method are shown in Table 2.

Table-2: Comparison of the results of $f(x) = e^{x} - 3x^{2} = 0$ at $x_0 = 0$ with different methods

Method	Number of Iterations	X_n	$f(X_n)$
Newton-Raphson	05	-0.458962274	−2.2E-08
Abbasbandy	05	-0.458964191	−6.5E-06
Basto	02	-0.458992962	−0.0001
Feng	05	-0.458962268	1.25E-10
Present	04	-0.458962268	−2.9E-11

Illustrative problem 3: Consider again the equation

$$f(x) = e^{x} - 3x^{2} = 0 \tag{9}$$

Taking $x_0 = 0.5$ as the initial approximation. Various results of different methods and the present method are shown in Table 3.

Table-3: Comparison of the results of $f(x) = e^{x} - 3x^{2} = 0$ at $x_0 = 0.5$ with different methods

Method	Number of Iterations	X_n	$f(X_n)$
Newton-Raphson	04	0.910007662	−2.7E-07
Abbasbandy	04	0.910007573	−1.5E-09
Basto	03	0.910007573	−1.5E-09
Feng	04	0.910007573	−6.3E-10
Present	04	0.910007572	−2.4E-12

Illustrative problem 4: Consider the following equation

$$f(x) = \sin x = 0 \tag{10}$$

The results obtained by Newton iteration, Hybrid iteration, Modified Newton method of Nasr-Al-Din Ide, Maheshwari's method and present iteration method are shown in Tables 4–8.

Table 4: Newton iteration for solving $f(x) = \sin x = 0$

n	x_n	$\|f(x_n)\|$
1	−12.601419947171719	0.03504215716101725900 (3.5E-2)
2	−12.566356255118672	0.00001435924050063514 (1.4E-5)
3	−12.566370614359174	0.00000000000000128651 (1.3E-15)

Table 5: Hybrid iteration for solving $f(x) = \sin x = 0$

n	x_n	$\|f(x_n)\|$
1	2.9869052804082314	0.15407121285051845000
2	3.2915361735771942	0.14938228643233631000
3	3.1467836457377532	0.00519096883489997360 (5.2E-3)
4	3.1410906419036979	0.00050201166500952066 (5.0E-4)
5	3.1415926228169786	0.00000003077281464126 (3.1E-8)
6	3.1415926525599063	0.00000000102988696493 (1.0E-9)
7	3.1415926545974751	0.00000000100768181543 (1.0E-9)
8	3.1415926576620539	0.00000000407226062837 (4.1E-9)
9	3.1415926674045491	0.00000001381475588335 (1.4E-8)
10	3.1415926662849540	0.00000001269516080004 (1.3E-8)
11	3.1415926605950051	0.00000000700521184360 (7.0E-9)
12	3.1415926640530745	0.00000001046328123000 (1.0E-8)
13	3.1415926498831088	0.00000000370668441530 (3.7E-9)
14	3.1415926445379925	0.00000000905180076483 (9.1E-9)
15	3.1415926440528610	0.00000000953693225536 (9.5E-9)
16	3.1415926443032149	0.00000000928657829558 (9.3E-9)
17	3.1415926408815600	0.00000001270823325495 (1.3E-8)

Table 6: Nasr-Al-Din Ide for solving $f(x) = \sin x = 0$

n	x_n	$\lvert f(x_n)\rvert$
1	−1.9296287714411738020	0.9363074853509102
2	−3.753451411476829580	0.5743899968181760
3	−3.118379573854536810	0.0232109950747530

Table 7: Maheshwari's method for solving $f(x) = \sin x = 0$

n	x_n	$\lvert f(x_n)\rvert$
1	−12.14025012250240000	0.413341334176536000
2	−12.56762318740160000	0.001252572714843290
3	−12.56637061435920000	0.000000000000000490 (4.9E-16)

Table 8: Present Method for solving $f(x) = \sin x = 0$

n	x_n	$\lvert f(x_n)\rvert$
1	3,147941963	−0.00635
2	3,14159269	−3.7E-08
3	3,141592654	1.23E-16

Different results for the equations of the form $f(x) = 0$ are shown in Table 1-8. Here, x_0 represents the initial approximation of the root of the equation $f(x) = 0$, n is the iteration number made to obtain the final result x_n and $f(x_n)$ represents the value of error at the nth iteration step w.r.t. its targeted value "0". It is apparent from these results that the present iterative formula is more effective than the other formulas of its class and gives the least value of working error. For instance, in illustrative problem-1, the present method gives $f(x_n) = 0$ in the 4th iteration step which clears that the solution obtained in this step is exact. Whereas, the other methods for the same example contains some error. Similarly, Table 2 & 3 for the other two examples shows the least value of error by the present method. Table 4-8 for the illustrative problem-2 also presents the stepwise results. The various results clears that the present method gives the least error as clear by the last columns under $\lvert f(x_n)\rvert$. This again proves the efficiency of the present method.

5. Conclusion

Various observations can be made from the result drawn in the Tables. The main conclusions are as follows:

I. Present method is a derivative free method that takes lesser number of iterations to obtain the required accuracy.

II. Various results in the last column of the Tables show the deviation in $f(x)$. Here, it can be concluded that the value of error is least in case of the present method.

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Mr. Amit Kumar Maheshwari is currently working at Advanced Materials and Processes Research Institute (AMPRI), CSIR, Bhopal- (Formerly RRL, BHOPAL) as CSIR-Senior Research Fellow. He has completed his M. Tech in Material Science in the year 2007 from Barkatullah University, Bhopal after doing M.Sc. in Mathematics & M.Sc. in physics. At present, he is pursuing Phd (Physics) from Barkatullah University, Bhopal. He has more than 5 years Research experience with a total of 21 publications in different international, national journals & conferences.