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A Note on a Multiplier Class

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Abstract Let f be a complex valued function on $[0, 2\pi]^N$, $\Lambda_1 = \left\{ f : \| T_\delta f - f \|_\infty = O\left(|\delta|\right)\right\}$, $\Lambda_1^1 = \left\{ f : \| T_\delta f - f \|_1 = O\left(|\delta|\right)\right\}$ and the multiplier class, $(L^\infty, \Lambda_1) = \{ f \in L^1([0,2\pi]^N) : f * g \in \Lambda_1, \forall g \in L^\infty([0,2\pi]^N\},$ where $\delta = (\delta_1, \delta_2, ..., \delta_N), T_\delta f(\mathrm{x}) = f(x+\delta), \forall \mathrm{x},\ \delta \in [0,2\pi]^N$. Here, we have characterized the class Λ_1^1 as $\Lambda_1^1 = (L^\infty, \Lambda_1)$.

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1. INTRODUCTION

The concept of generalized bounded variations, Lipschitz condition, absolute continuity, etc., are generalized to integral setting as well as from one variable to several variables to obtain various classes of functions which are of interest in Functional Analysis as well as in Fourier analysis (Avdispahic, 1986; James Caveny, 1970; De Leeuw, 1961; Ghorpade and Limaye, 2010; Vyas and Darji, 2012). Lipschitz conditions and integrated Lipschitz conditions for single variable functions as well as for several variables functions give classes which are important for the discussion of the convergence, uniform convergence and absolute convergence of single Fourier series as well as of multiple Fourier series respectively. Objective of the paper is to obtain some relationship between the Lipschitz classes and the integrated Lipschitz classes of functions of several variables.

First, recall some standard notations for the function classes of interest.

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2. NOTATIONS AND DEFINITIONS

In the sequel, f is a complex valued function of N-variables defined over $[0,\ 2\pi]^N$, which is 2π-periodic in each variable, $\delta = (\delta_1, ...\delta_N), \| \delta \|_q =$ $\left(\sum_{i=1}^N |\delta_i|^q\right)^{1/q} (1 \le q \le \infty)$ and

$$(\Delta_{\mathrm{h}} f)(\mathrm{x}) = (T_{(\mathrm{h})} f - f)(\mathrm{x})$$

$$= \sum_{\eta_1=0}^{1} \cdots\cdots \sum_{\eta_N=0}^{1} (-1)^{\eta_1+\cdots+\eta_N} f(x_1+\eta_1 h_1, x_2+\eta_2 h_2, \ldots x_N+\eta_N h_N) \ , \quad (2.1)$$

where $(T_{\mathbf{h}}f)(\mathrm{x}) = f(\mathrm{x} + \mathbf{h})$ for all $\mathrm{x} = (x_1, x_2, \ldots\, x_N)$, $\mathbf{h} = (h_1, h_2, \ldots, h_N) \in [0, 2\pi]^N$.

For any h and δ define $h \le \delta$ if and only if $h_i \le \delta_i$ for all $i = 1$ to N.

$L^1([0, 2\pi]^N)$, is the space of complex valued Lebesgue integrable functions over $[0, 2\pi]^N$ which are 2π periodic in each variables and $L^\infty([0, 2\pi]^N)$ consists of the $L^1([0, 2\pi]^N)$ functions which are essentially bounded. $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denote the norms in the spaces $L^1([0, 2\pi]^N)$ and $L^\infty([0, 2\pi]^N)$ respectively.

$$\Lambda_1\left([0,2\pi]^N\right) = \left\{ f \varepsilon L^\infty\left([0,2\pi]^N\right) : \sup_{0 \le \mathrm{h} \le \delta} \| T_{\mathrm{h}} f - f \|_\infty = O\left(\| \delta \|_2\right)\right\}.$$

$$\Lambda_1^1\left([0,2\pi]^N\right) = \left\{ f \varepsilon L^1\left([0,2\pi]^N\right) : \sup_{0 \le \mathrm{h} \le \delta} \| T_{\mathrm{h}} f - f \|_1 = O\left(\| \delta \|_2\right)\right\}.$$

$$Lip\left([0,2\pi]^N\right) = \left\{ f \varepsilon L^\infty\left([0,2\pi]^N\right) : \sup_{0 \le \mathrm{h} \le \delta} \| T_{\mathrm{h}} f - f \|_\infty = O\left(\| \delta \|_1\right)\right\}.$$

$$Lip^1\left([0,2\pi]^N\right) = \left\{ f \varepsilon L^1\left([0,2\pi]^N\right) : \sup_{0 \le \mathrm{h} \le \delta} \| T_{\mathrm{h}} f - f \|_1 = O\left(\| \delta \|_1\right)\right\}.$$

$$\Lambda_{11}\left([0,2\pi]^N\right) = \left\{ f \varepsilon L^\infty\left([0,2\pi]^N\right) : \sup_{0 \le \mathrm{h} \le \delta} \| \Delta_{\mathrm{h}} f \|_\infty = O\left(\| \delta \|_2\right)\right\}.$$

$$Lip_{11}\left([0,2\pi]^N\right) = \left\{ f \varepsilon L^1\left([0,2\pi]^N\right) : \sup_{0 \le \mathrm{h} \le \delta} \| \Delta_{\mathrm{h}} f \|_\infty = O\left(\| \delta \|_1\right)\right\}.$$

$$Lip_{11}^1\left([0,2\pi]^N\right) = \left\{ f \varepsilon L^1\left([0,2\pi]^N\right) : \sup_{0 \le \mathrm{h} \le \delta} \| \Delta_{\mathrm{h}} f \|_1 = O\left(\| \delta \|_1\right)\right\}.$$

$$Lip_{*11}\left([0,2\pi]^N\right) = \{ f \varepsilon Lip_{11}\left([0,2\pi]^N\right) :$$

$$\sup_{h_i \in (0,\delta_i]} \| \Delta_{(0,\ldots,0,h_i,0,\ldots 0)} f \|\infty = O(|\delta_i|), \quad \forall\, i = 1 \text{ to } N\}$$

$$Lip_{*11}^1\left([0,2\pi]^N\right) = \{ f \in Lip_{11}^1\left([0,2\pi]^N\right) :$$

$$\sup_{h_i \in (0,\delta_i]} \| \Delta_{(0,\ldots,0,h_i,0,\ldots 0)} f \|_1 = O(|\delta_i|), \quad \forall\, i = 1 \text{ to } N$$

For any $f, g \in L^1([0, 2\pi]^N)$ their convolution product $f * g$ is the $L^1([0, 2\pi]^N)$ function defined (Vyas, 2013) as

$$(f * g)(\mathrm{x}) = \frac{1}{(2\pi)^N} \int_{[0,2\pi]^N} f(\mathrm{x}-\mathrm{t}) g(\mathrm{t}) d\mathrm{t}, \quad \forall \mathrm{x} \in [0,2\pi]^N.$$

In general the multiplier class is defined as

$$(L^{\infty}[0,2\pi]^N, E) = \{f \in L^1([0,2\pi]^N) : f * g \in E \text{ for all } g \in L^{\infty}([0,2\pi]^N)\},$$

for $E = \Lambda_1([0,2\pi]^N)$, or $E = Lip\ ([0,2\pi]^N)$, or $E = Lip_{11}\ ([0,2\pi]^N)$, or $E = Lip_{*11}([0,2\pi]^N)$.

Here, we prove the following results for functions of several variables.

Theorem 2.1. $\Lambda_1^1([0,2\pi]^N) = (L^{\infty}([0,2\pi]^N), \Lambda_1([0,2\pi]^N))$.

Theorem 2.2. $Lip^1([0,2\pi]^N) = (L^{\infty}([0,2\pi]^N), Lip([0,2\pi]^N))$.

Theorem 2.3. $Lip_{*11}^1([0,2\pi]^N) = (L^{\infty}([0,2\pi]^N), Lip_{*11}([0,2\pi]^N))$.

For the simplicity of proof, we will prove all these results for N = 2.

Proof of the Theorem 2.1. For any $f \in \Lambda_1^1([0,2\pi]^2), g \in L^{\infty}([0,2\pi]^2)$ and **x, y** $\in ([0,2\pi]^2)$, we have

$$\begin{aligned}
&\left|(f*g)(\mathrm{x})-(f*g)(\mathrm{y})\right| \\
&\qquad \le \frac{1}{(2\pi)^2}\int_{[0,2\pi]^2}\left|f(\mathrm{x}-\mathrm{t})-f(\mathrm{y}-\mathrm{t})\right|\left|g(\mathrm{t})\right|d\mathrm{t} \\
&\qquad \le \frac{\|g\|_{\infty}}{(2\pi)^2}\int_{[0,2\pi]^2}\left|f(\mathrm{x}-\mathrm{t})-f(\mathrm{y}-\mathrm{t})\right|d\mathrm{t} = O(\|\mathrm{x}-\mathrm{y}\|_2).
\end{aligned}$$

Now suppose that $f \in (L^{\infty}([0,2\pi]^2), \Lambda_1([0,2\pi]^2))$ The Lipschitz class $\Lambda_1([0,2\pi]^2)$ **i**s a Banach space with respect to the norm

$$\|h\| = \left|h(0,0)\right| + \sup_{\mathrm{x},\mathrm{y}\in[0,2\pi]^2}\frac{\left|h(\mathrm{y})-h(\mathrm{x})\right|}{\|\mathrm{y}-\mathrm{x}\|_2}.$$

Define the map

$$T : L^{\infty}([0,2\pi]^2) \to \Lambda_1([0,2\pi]^2) \ \ as \ \ T(g) = f * g,\ g \in L^{\infty}([0,2\pi]^2)$$

To prove that T is continuous it is enough to show that the graph of T is closed.

Consider $g_n \to g$ in $L^{\infty}([0,2\pi]^N)$ and $T(g_n) \to h$ in $\Lambda_1([0,2\pi]^2)$ then for any fixed **x** $\in ([0,2\pi]^2)$, we have

$$\begin{aligned}
\left|(T(g))(\mathrm{x})-h(\mathrm{x})\right| &\le \left|(f*g)(\mathrm{x})-(f*g_n)(\mathrm{x})\right| + \left|(T(g_n))(\mathrm{x})-h(\mathrm{x})\right| \\
&\le \|f\|_1\|g_n-g\|_{\infty} + \|T(g_n)-h\|.
\end{aligned}$$

Hence, $T(g) = h$ implies T is continuous and

$$\| T(g) \| \le \| T \| \| g \|_\infty, \ \forall g \in L^\infty([0,2\pi]^2), \textit{ where } \| T \| \textit{ is a constant}$$

Thus, for any **x, y** $\in ([0,2\pi]^2)$, we have

$$\left| \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} (f(\mathrm{x}-\mathrm{t}) - f(\mathrm{y}-\mathrm{t}))\, g(\mathrm{t})\, d\mathrm{t} \right| \le \| T \| \| g \|_\infty \| \mathrm{x}-\mathrm{y} \|_2 .$$

In the above inequality for $g(\mathrm{t}) = e^{\{-i\, arg(f(\mathrm{x}-\mathrm{t}) - f(\mathrm{y}-\mathrm{t}))\}}$, we have

$$\frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} \left| f(\mathrm{x}-\mathrm{t}) - f(\mathrm{y}-\mathrm{t}) \right| dt \le \| T \| \| \mathrm{x}-\mathrm{y} \|_2 .$$

This completes the proof.

Similarly, we can prove the Theorem 2.2.

By using the fact that; $Lip_{*11}([0,2\pi])$ is a Banach space with respect to the norm

$$\| h \| = \left| h(0,0) \right| + \sup_{\mathrm{x,y} \in [0,2\pi]^2} \frac{\left| h(x_1,x_2) - h(y_1,x_2) - h(x_1,y_2) + h(y_1,y_2) \right|}{\left| y_1 - x_1 \right| + \left| y_2 - x_2 \right|}$$

$$+ \sup_{\mathrm{x,y} \in [0,2\pi]^2} \frac{\left| h(x_1,x_2) - h(y_1,x_2) \right|}{\left| y_1 - x_1 \right|} + \sup_{\mathrm{x,y} \in [0,2\pi]^2} \frac{\left| h(x_1,x_2) - h(x_1,y_2) \right|}{\left| y_2 - x_2 \right|}$$

where **x** = (x_1, x_2) and **y** = (y_1, y_2).

Similarly one can prove the Theorem 2.3.

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