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Consolidation of a Poroelastic Half–Space

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Abstract A porous medium is an elastic solid permeated by an interconnected network of pores filled with a fluid. Both solid and pore network are assumed to be continuous so as to form two interpenetrating continua. The theory of poroelasticity investigates the time-dependent coupling between the deformation of the elastic solid skeleton and fluid flow within the skeleton. The solid-to-fluid and fluid-to-solid couplings are assumed to occur instantaneously in the quasi-static approximation in which elastic wave propagation is ignored. Consolidation of a poroelastic body takes place when it is acted upon by surface loads. The study of consolidation of a poroelastic half-space or stratum has received much attention due to its geophysical and engineering applications. The aim of the present paper is to review recent work on the subject, indicating the assumptions made, methods used and conclusions drawn.

Keywords: anisotropic permeability; consolidation; half-space; multi-layered; plane strain; poroelastic; quasi-static; surface loads.

1. INTRODUCTION

The earliest theory to account for the influence of pore fluid on the quasi-static deformation of soils was developed by Terzaghi (1923), who proposed a model of one-dimensional consolidation. Biot (1941) was the first to develop the three-dimensional theory of poroelasticity. Subsequently, Biot (1955, 1956 a,b) extended his theory to consider the effect of anisotropy and wave propagation in fluid-filled porous media. Rice and Cleary (1976) reformulated Biot's linear constitutive equations and replaced the new elastic constants introduced by Biot with more familiar constants (Poisson's ratio and bulk modulus) evaluated in both the drained and the undrained states.

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They solved a number of geophysical problems, including that of a suddenly introduced edge dislocation, concentrated line force and suddenly pressurized cylindrical and spherical cavities. Biot's theory has been used very extensively for studying the consolidation of a poroelastic medium.

McNamee and Gibson (1960a) have shown that the task of determining the displacements and stresses in a poroelastic medium could be facilitated by the introduction of two displacement functions. Using these functions, McNamee and Gibson (1960b) solved plane strain and axially-symmetric consolidation problems of a semi-infinite clay stratum having incompressible fluid and solid constituents with isotropic permeability. Two problems of a semi-infinite body to the surface of which a uniform pressure is applied along an infinite strip or over a circular area were solved. Schiffman and Fungaroli (1965) extended the displacement function formulation to non-axisymmetric problems. They studied consolidation of a semi-infinite solid subjected to a uniform tangential load at a pervious and an impervious surface using three displacement functions.

Bell and Nur (1978) used two-dimensional half-space models with surface loading to study the change produced by reservoir-induced pore pressure and stresses for thrust, normal and strike-slip faults. Normally the elastic stresses and pore pressure influence each other in a porous medium. However, they simplified the analysis by assuming that only stresses influence the pore pressure and not vice-versa. Booker and Randolph (1984) discussed the consolidation of a poroelastic half-space with cross-anisotropic deformation and flow properties using integral transforms. They studied the effect of circular and rectangular loading numerically. The stress and pore pressure changes produced by a steady periodic variation of water level on the surface of a uniform porous elastic half-space were evaluated by Roeloffs (1988), using coupled Biot equations of elastic deformation and pore fluid flow. Yue and Selvadurai (1995) examined the axisymmetric interaction between a rigid, circular, flat indentor and a poroelastic half-space. Three drainage conditions (completely drained, partially drained or completely undrained) at the surface of the poroelastic half-space were considered. Kalpna and Chander (1997) obtained stresses and pore pressure for an impervious elastic layer resting on a water-saturated porous elastic half-space when the upper surface of the layer is acted upon by a normal stress field varying harmonically in time. In a subsequent study, Kalpna and Chander (2000) calculated stresses and pore pressure in a porous elastic half-space for a time-varying finite reservoir surface load using Green's function approach.

Mei *et al.* (2004) presented a finite layer procedure for Biot's consolidation analysis of layered soil using a cross-anisotropic elastic constitutive model.

Both fluid and solid constituents were assumed to be incompressible. The immediate settlement, the final settlement and consolidation behaviour of a square footing were studied. Chen (2005) discussed the steady-state response of a multilayered poroelastic half-space to a point sink. Both the permeability and the poroelasticity of the medium were assumed to be transversely isotropic, but its fluid and solid constituents were assumed to be incompressible. Chen *et al.* (2005a) studied axisymmetric consolidation of a semi-infinite, transversely isotropic saturated soil subjected to a uniform circular loading at the ground surface.

Singh and Rani (2006) solved two-dimensional plane strain problem of the quasi-static deformation of a multi-layered poroelastic half-space by surface loads. The stresses and pore pressure were taken as basic state variables. Both fluid and solid constituents were assumed to be compressible with isotropic permeability. Conte (2006) presented the analysis of coupled consolidation in unsaturated soil under the condition of plane strain as well as axial symmetry due to strip and circular loads. Fourier transform method for plane strain and Hankel transform method for axially-symmetric problem have been employed. Singh *et al.* (2007) discussed the quasi-static plane strain deformation of a poroelastic half-space with anisotropic permeability and compressible constituents by two-dimensional surface loads. An analytical solution was obtained by using a pure compliance formulation. Biot's stress function was used to decouple the governing equations. The problem of normal strip loading was discussed in detail. In a subsequent study, Singh *et al.* (2009) investigated the problem of the consolidation of a poroelastic half-space with anisotropic permeability and compressible fluid and solid constituents by axisymmetric surface loads.

Ai *et al.* (2008) solved Biot's three-dimensional consolidation problem for a saturated poroelastic multi-layered soil due to loading at an arbitrary interface in the Cartesian coordinate system using transfer matrix method. The corresponding problem of circular loading has been discussed by Ai *et al.* (2010a). Subsequently, Ai *et al.* (2010b) presented transfer matrix solutions to study the axisymmetric and non-axisymmetric consolidation of a multi-layered soil under arbitrary loading. However, in all the above three studies, the medium is assumed to be incompressible and permeability isotropic.

The assumption of modeling the medium as a half-space is applicable only to consolidation problems where the thickness of the soil stratum is much greater than the dimensions of the loaded area. In other cases, it is necessary to model the poroelastic medium as a finite layer. Gibson et al. (1970) obtained the solution for consolidation of a uniform clay layer resting on a smooth-rigid base subjected to circular or strip loading. Booker (1974) presented a solution to the problem of the consolidation of a uniform clay layer subjected to general normal surface loading with the assumption that the lower surface of the strip adheres completely to a rigid base. Solutions for the case of uniformly loaded strip, circle and square were evaluated for a variety of Poisson's ratio. However, it was assumed that the solid and fluid constituents were incompressible and permeability isotropic. Booker and Small (1987) presented a method for obtaining the consolidation of a layered soil subjected to strip, circular, or rectangular surface loading, or subjected to fluid withdrawal due to pumping by using direct numerical inversion of Laplace transforms.

Using Biot's theory of poroelasticity, Yue et al. (1994) presented an analytical investigation of the quasi-static development of excess pore pressure in a poroelastic seabed layer. The layer was modelled as a poroelastic medium of finite thickness, saturated with a compressible pore fluid and resting on a rough-rigid impermeable base. In a subsequent study, Selvadurai and Yue (1994) examined the axisymmetric contact problem related to the indentation of a fluid-saturated poroelastic layer resting on a rigid impermeable base due to circular foundation. Conte (1998) presented a numerical procedure to analyze consolidation problem involving anisotropic layered soils which contain incompressible as well as compressible pore fluid caused by surface loading.

Chen *et al.* (2005b) presented a semi-analytical solution to axisymmetric consolidation of a transversely isotropic soil layer resting on a rough impervious base and subjected to a uniform circular pressure at the surface. The medium was assumed to be transversely isotropic in its elastic and hydraulic properties. However, in numerical computations, only the effect of the elastic anisotropy was studied. An analytical solution for the consolidation of a soil layer subjected to vertical point loading was presented by Chen *et al.* (2007). The medium was assumed to have incompressible fluid and solid constituents with isotropic permeability. The axisymmetric consolidation problem of a finite soil layer has been studied by Ai and Wang (2008). A solution for plane strain consolidation of a soil layer with anisotropic permeability and incompressible fluid and solid constituents due to surface loads was obtained by Ai and Wu (2009). Rani *et al.* (2011) obtained the corresponding axisymmetric solution when the fluid and solid constituents are compressible.

Closed-form solutions for the steady-state distribution of displacement, pore-pressure and stress around a point sink have been obtained by Booker and Carter (1986). These solutions have been obtained for the long-term settlement caused by withdrawal of fluid from a point sink at finite depth below the surface of a homogeneous isotropic porous elastic half-space with isotropic permeability. Booker and Carter (1987a) presented a solution for the transient effect of pumping fluid from a point sink embedded in a saturated, porous, elastic half-space. They assumed that the medium is homogeneous and isotropic with respect to its elastic properties, anisotropic with respect to flow of the pore fluid which was considered to be incompressible. The corresponding problem of withdrawal of a compressible pore fluid from a point sink in an isotropic elastic half-space with anisotropic permeability has been discussed by Booker and Carter (1987b).

Tarn and Lu (1991) presented analytical solutions of the long-term consolidation and excess pore water pressure due to fluid withdrawal from a saturated porous elastic half-space. In their analysis, both permeability and elastic properties were considered to be cross-anisotropic. Chau (1996) obtained fundamental solutions for the interior fluid point source and point forces embedded in a poroelastic half-space with incompressible constituents and isotropic permeability. Ganbe and Kurashige (2000) obtained fundamental solutions for an elastically isotropic poroelastic solid having transversely isotropic permeability due to instantaneous fluid point source and instantaneous point force by using Laplace-Fourier transform method. Taguchi and Kurashige (2002) obtained fundamental solutions for point forces acting on three orthogonal directions and an instantaneous fluid point source in a fluid-saturated, porous, infinite solid of transversely isotropic elasticity and permeability. These solutions are in explicit form but quite lengthy.

Wang and Kuumpel (2003) presented a numerical scheme to compute poroelastic solutions for excess pore pressure and displacements in a multi-layered half-space using mirror-image technique and an extension of Haskell's propagator method. Using propagator matrix technique, Chen (2003) presented analytical solutions for the steady-state response of a multi-layered poroelastic half-space subjected to pumping. Chen and Gallipoli (2004) derived an analytical solution for the steady state infiltration from a buried point source into a heterogeneous cross-anisotropic unsaturated half-space. Lu and Hanyga (2005) obtained fundamental solution for a layered porous half-space subjected to a vertical point force or a point source. Singh and Rani (2007) formulated the two-dimensional plane strain problem of the quasi-static deformation of a multi-layered poroelastic half-space with compressible constituents by internal sources. Pure compliance approach is used to formulate the problem. The integral expressions for the surface displacement and fluid flux are obtained for a vertical line force, a horizontal line force and a fluid injection line source using propagator matrix approach.

Further references to studies on the application of Biot's theory of poroelasticity to consolidation problems can be found in Detournay and Cheng (1993), Wang (2000), Rudnicki (2001) and Singh (2005).

2. CONSTITUTIVE EQUATIONS

The four basic variables of poroelasticity are: stress, strain, pore pressure and increment of fluid content. Stress and strain are symmetric second-order tensors and, therefore, can be represented by six independent components each. Pore pressure and increment of fluid content are scalar quantities. Thus, there are fourteen variables in all:

$\sigma ij = \sigma ji$ (six independent components of stress),
$\in_{ij} = \in_{ji}$ (six independent components of strain),
p (increase in pore fluid pressure), and
ζ (increment of fluid content).

We may take $\in_{ij}$, ζ as the seven independent variables and σ_{ij}, p as the seven dependent variables, or, vice-versa. Biot's constitutive equations consist of a set of seven linear homogeneous equations expressing the seven dependent variables in terms of the seven independent variables. Four poroelastic constants occur in the seven constitutive equations for an isotropic poroelastic material. The set of constitutive equations in which the stress is considered as a dependent variable is called the stiffness formulation. In contrast, the set of constitutive equations in which the strain is considered as a dependent variable is called the compliance formulation.

In the compliance formulation, the constitutive equations for an isotropic poroelastic body can be expressed in the form (Wang, 2000)

$$\in_{ij} = \frac{1}{2G}\left[\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{mm}\delta_{ij} + \frac{1-2\nu}{1+\nu}\alpha p\delta_{ij}\right], \tag{2.1}$$

where G is the shear modulus, ν the drained Poisson's ratio, α the Biot-Willis coefficient and δ_{ij} the Kronecker delta. Further,

$$\zeta = \frac{\alpha}{K}\left(\frac{\sigma_{mm}}{3}\right) + \frac{\alpha}{KB}p, \tag{2.2}$$

where K is the drained bulk modulus and B the Skempton's constant. Equations (2.1) and (2.2) constitute the complete set of seven constitutive equations in the compliance formulation. Four poroelastic constants (e.g. G, ν, α, B) are needed for the complete characterization of the poroelastic material (K can be expressed in terms of G and ν).

Equations (2.1) and (2.2) can be inverted to obtain the constitutive equations in the stiffness formulation. We find

$$\sigma_{ij} = 2G\left[\in_{ij} + \frac{\nu_u}{1-2\nu_u}\in_{mm}\delta_{ij} - \frac{B(1+\nu_u)}{3(1-2\nu_u)}\zeta\delta_{ij}\right], \tag{2.3}$$

$$\zeta = \alpha\in_{mm} + \frac{\alpha}{K_u B}p, \tag{2.4}$$

where ν_u is the undrained Poisson's ratio and K_u the undrained bulk modulus defined by the relations

$$\nu_u = \frac{3\nu + \alpha B(1\text{-}2\nu)}{3 - \alpha B(1-2\nu)},$$

$$K_u = \frac{2(1+\nu_u)G}{3(1-2\nu_u)}.$$

We can take (G, ν, ν_u, B) as the four independent poroelastic constants instead of (G, ν, α, B). Detournay and Cheng (1993) chose (G, ν, ν_u, B) as the four independent poroelastic constants. In fact, any four of the seven constants (G, ν, ν_u, K, K_u, α, B) can be chosen as the four independent poroelastic constants and the remaining three constants can be expressed in terms of these four constants. Some useful relations among various poroelastic constants are given by Singh (2005). For incompressible solid constituents of the poroelastic material, $\alpha = 1$ (Detournay & Cheng, 1993; Wang, 2000). For incompressible solid and fluid constituents of the poroelastic material, $\alpha = 1$, $\nu_u = 0.5$.

Putting the value of ζ from (2.4) into equation (2.3), we obtain

$$\sigma_{ij} = 2G\left[\in_{ij} + \frac{\nu}{1-2\nu}\in_{mm}\delta_{ij}\right] - \alpha p\delta_{ij}. \tag{2.5}$$

Similarly, from equations (2.1) and (2.2), we obtain

$$\in_{ij} = \frac{1}{2G}\left[\sigma_{ij} - \frac{\nu_u}{1+\nu_u}\sigma_{mm}\delta_{ij}\right] + \frac{1}{3}B\zeta\delta_{ij}. \tag{2.6}$$

The importance of poroelastic coupling in a physical situation depends upon the rate of pore fluid flow relative to the change in the stress conditions. The drained and undrained cases are the limiting cases of the slow and fast loading, respectively. Relatively slow loading leaves the pore pressure unchanged in the control volume because fluid flow has adequate time to equilibrate with an external boundary. In contrast, little fluid flows into or out of the control volume if the loading is rapid.

3. FIELD EQUATIONS

For a fluid-saturated poroelastic medium, there are eleven unknown field variables: six stress components ($\sigma_{ij} = \sigma_{ji}$), three displacement components (u_i), pore pressure (p) and increment of fluid content (ζ). The eleven field variables are to be determined by solving eleven field equations: three equilibrium equations, one fluid diffusion equation, six constitutive equations for stress and one constitutive equation for pore pressure. The fluid diffusion equation is obtained by combining Darcy's law of fluid flow with the equation of continuity and involves first time-derivative of stress. Consequently, the solution is time-dependent. Since we are considering equilibrium equations rather than the equations of motion, the solution obtained will be quasi-static.

3.1. Equilibrium Equations

$$\frac{\partial \sigma_{ji}}{\partial x_j} + F_i = 0, \tag{3.1}$$

where **F** (F_1, F_2, F_3) is the body force per unit volume. Using the constitutive equations given in Section 2 in the equation of equilibrium (3.1), we obtain

$$\nabla^2 \left(\sigma_{ii} + 4\eta p\right) + \left(\frac{1+\nu}{1-\nu}\right)\frac{\partial F_i}{\partial x_i} = 0 \tag{3.2}$$

where

$$\eta = \frac{1-2\nu}{2(1-\nu)}\alpha \left(0 \le \eta \le \frac{1}{2}\right) \tag{3.3}$$

is the *poroelastic stress coefficient*. Equation (3.2) is known as the *compatibility equation*.

3.2. Fluid Diffusion Equation

The increment of fluid content ζ is the volume of the fluid imported into a control volume per unit control volume. Fluid flux **q** is the volume of fluid crossing per unit area of the control volume per unit time. Therefore, the equation of continuity can be expressed in the form

$$\frac{\partial \zeta}{\partial t} = -\mathrm{div}\ \mathbf{q}. \tag{3.4}$$

The negative sign is due to the sign convention that ζ is positive for fluid entering the control volume and **q** is positive for fluid leaving the control volume.

According to Darcy's law of fluid, flow in a fluid-saturated isotropic porous medium

$$\mathbf{q} = -\chi \text{ grad p}, \tag{3.5}$$

where χ is the *Darcy conductivity*. From equations (3.4) and (3.5), we obtain the fluid diffusion equation

$$\frac{\partial \zeta}{\partial t} = \chi \nabla^2 p. \tag{3.6}$$

Putting the value of ζ from the constitutive equation (2.2) into equation (3.6) and using equation (3.2) for zero body force, we obtain

$$\left(c\nabla^2 - \frac{\partial}{\partial t}\right)\left(\sigma_{mm} + \frac{3}{B}p\right) = 0 \tag{3.7}$$

where

$$c = \frac{2GB^2(1-\nu)(1+\nu_u)^2 \chi}{9(1-\nu_u)(\nu_u - \nu)} \tag{3.8}$$

is the *hydraulic diffusivity* or *consolidation coefficient*. It is directly proportional to the rock permeability.

We note that five poroelastic constants occur in the field equations for an isotropic poroelastic medium. Out of these, four constants are introduced by the constitutive equations. The fifth constant (χ) is introduced by Darcy's law. Using equation (3.8), we may choose c to be the fifth constant instead of χ.

4. PLANE STRAIN CONSOLIDATION OF AN ISOTROPIC POROELASTIC HALF-SPACE

For plane strain deformation of a poroelastic medium in the x_1x_3-plane, the displacement components are of the form

$$u_1 = u_1(x_1, x_3, t),\ u_2 = 0,\ u_3 = u_3(x_1, x_3, t). \tag{4.1}$$

The constitutive equation (2.1) yields

$$\begin{aligned}
&2G \in_{11} = (1-\nu)\sigma_{11} - \nu\sigma_{33} + \alpha_0 p,\\
&2G \in_{33} = (1-\nu)\sigma_{33} - \nu\sigma_{11} + \alpha_0 p,\\
&2G \in_{13} = \sigma_{13},\\
&\sigma_{21} = \sigma_{23} = 0,\\
&\sigma_{22} = \nu(\sigma_{11} + \sigma_{33}) - \alpha_0 p,
\end{aligned} \tag{4.2}$$

where

$$\alpha_0 = (1-2\nu)\alpha.$$

Similarly, the compatibility equation (3.2) becomes

$$\nabla^2(\sigma_{11} + \sigma_{33} + 2\eta p) = 0, \tag{4.3}$$

assuming zero body force. The diffusion equation (3.7) is replaced by

$$\left(c\nabla^2 - \frac{\partial}{\partial t}\right)\left[\sigma_{11} + \sigma_{33} + \frac{3}{(1+\nu_u)B}p\right] = 0. \tag{4.4}$$

Equations of equilibrium (3.1) reduce to

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{13}}{\partial x_3} = 0, \quad \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{33}}{\partial x_3} = \tag{4.5}$$

and equation (2.2) becomes

$$\zeta = \frac{\alpha(1+\nu)}{3K}\left[\sigma_{11} + \sigma_{33} + \frac{3}{B(1+\nu_u)}p\right]. \tag{4.6}$$

The coupled system of equations (4.3) to (4.5) can be solved in terms of Biot's stress function F (Roellofs, 1988; Wang, 2000):

$$\sigma_{11} = \frac{\partial^2 F}{\partial x_3^2}, \quad \sigma_{33} = \frac{\partial^2 F}{\partial x_1^2}, \quad \sigma_{13} = -\frac{\partial^2 F}{\partial x_1 \partial x_3} \tag{4.7}$$

The equilibrium equations (4.5) are then identically satisfied. Equations (4.3), (4.4) and (4.7) yield

$$\nabla^2\left(\nabla^2 F + 2\eta p\right) = 0, \tag{4.8}$$

$$\left(c\nabla^2 - \frac{\partial}{\partial t}\right)\left[\nabla^2 F + \frac{3}{(1+\nu_u)B}p\right] = 0. \tag{4.9}$$

Eliminating F and p in turn, equations (4.8) and (4.9) lead us to the following decoupled equations in p and F

$$\left(c\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2 p = 0, \tag{4.10}$$

$$\left(c\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^4 F = 0. \tag{4.11}$$

The general solution of equations (4.10) and (4.11) can be obtained by taking Laplace transform of these equations with respect to time and then solving the resulting equations in space variables. The stresses then follow from equation (4.7). Finally, the displacements can be obtained from equation (4.2) on expressing strains in terms of displacements and integrating the resulting equations. After a lengthy algebra, we obtain (Singh and Rani, 2006)

$$p = -\int_0^\infty \left[\frac{s}{2c\eta}\left(B_1 e^{-mz} + D_1 e^{mz}\right) + \xi k^2 \left(-B_3 e^{-kz} + D_3 e^{kz}\right)\right] \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} dk \tag{4.12}$$

$$q = \chi \int_0^\infty \left[\frac{ms}{2c\eta}\left(-B_1 e^{-mz} + D_1 e^{mz}\right) + \xi k^3 \left(B_3 e^{-kz} + D_3 e^{kz}\right)\right] \begin{pmatrix} \sin & kx \\ \cos & kx \end{pmatrix} dk, \tag{4.13}$$

$$\sigma_{11} = \int_0^\infty \Big[m^2 \left(B_1 e^{-mz} + D_1 e^{mz}\right) + k^2 \left(B_2 e^{-kz} + D_2 e^{kz}\right)$$
$$+ k^2 \left\{ (kz-2) B_3 e^{-kz} + (kz+2) D_3 e^{kz} \right\}\Big] \begin{pmatrix} \sin & kx \\ \cos & kx \end{pmatrix} dk, \tag{4.14}$$

$$\sigma_{33} = -\int_0^\infty \left[B_1 e^{-mz} + D_1 e^{mz} + (B_2 + B_3 kz) e^{-kz} + (D_2 + D_3 kz) e^{kz} \right] \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} k^2 dk \tag{4.15}$$

$$\sigma_{13} = \int_0^\infty \Big[m\left(B_1 e^{-mz} - D_1 e^{mz}\right) + k\left(B_2 e^{-kz} - D_2 e^{kz}\right)$$
$$+ \ k\left\{ B_3 (kz-1) e^{-kz} - D_3 (kz+1) e^{kz} \right\}\Big] \begin{pmatrix} \cos & kx \\ \text{-}\sin & kx \end{pmatrix} k\,dk, \tag{4.16}$$

$$2Gu_1 = -\int_0^\infty \Big[B_1 e^{-mz} + D_1 e^{mz} + B_2 e^{-kz} + D_2 e^{kz} + B_3 \left(2\nu_u - 2 + kz\right) e^{-kz}$$
$$+ D_3 \left(-2\nu_u + 2 + kz\right) e^{kz} \Big] \begin{pmatrix} \cos kx \\ \text{-}\sin kx \end{pmatrix} k\, dk \tag{4.17}$$

$$2Gu_3 = \int_0^\infty \Big[m\left(B_1 e^{-mz} - D_1 e^{mz}\right) + k\left(B_2 e^{-kz} - D_2 e^{kz}\right) + B_3 \left(1 - 2\nu_u + kz\right) k e^{-kz}$$
$$+ D_3 \left(1 - 2\nu_u - kz\right) k e^{kz} \Big] \begin{pmatrix} \sin & kx \\ \cos & kx \end{pmatrix} dk, \tag{4.18}$$

where

$$\xi = \frac{2}{3}(1+\nu_u) B, \qquad m = \left(k^2 + \frac{s}{c}\right)^{1/2}, \tag{4.19}$$

$z = x_3$, q is the fluid flux in the z-direction and s is the Laplace transform variable.

We have found the solution in the Fourier-Laplace domain involving six arbitrary constants (B_1, D_1, B_2, D_2, B_3, D_3). These constants are to be found from the boundary conditions. Two integrations are required to be performed to get the solution in the space-time domain. The first integration is over the wave number k and can be evaluated, for example, by applying the Guass quadrature. The second integration is the inverse Laplace transform and can be evaluated by following one of the various numerical schemes available.

The plane strain solution of Biot's coupled system of deformation-diffusion equations given by equations (4.12)-(4.18) can be used to find solutions of various problems involving boundaries parallel to the plane $z = 0$. Singh and Rani (2006) used this solution to study analytically the plane strain deformation of a multi-layered poroelastic half-space by surface loads using generalized Thomson-Haskell matrix method. Explicit expressions of the elements of the propagator matrix are given. Pan (1999) used Green's function method to solve the consolidation problem of a multi-layered poroelastic half-space. However, the elements of the propagator matrix given by him are complicated functions of poroelastic constants and some of the elements are incorrect. Wang and Fang (2003) also studied the consolidation of a multi-layered poroelastic half-space. However, only three poroelastic constants appear in their formulation as against five constants which define a general homogeneous isotropic poroelastic medium. Five poroelastic constants appear in our formulation.

5. CONSOLIDATION OF A POROELASTIC HALF-SPACE WITH ANISOTROPIC PERMEABILITY

Consolidation of a poroelastic half-space by surface loads has been studied extensively. However, in most of the investigations, the hydraulic permeability is assumed to be isotropic. Permeability determines the ability of the porous medium to conduct fluid flow in its pores and, therefore, can be different in different directions. In most cases, the soil deposits are the result of a sedimentation process that produces horizontal stratification planes. Consequently, permeability in the horizontal and vertical directions may differ. For important geophysical and engineering applications, it is useful to study the effect of anisotropy in permeability on the consolidation of a half-space by surface loads.

Plane strain consolidation of a poroelastic half-space possessing isotropic permeability and compressible fluid and solid constituents has been considered in the last section. In this section, we study the corresponding problem when the permeability is anisotropic. It may be noted that the constitutive equations (4.2), the compatibility equation (4.3) and the equilibrium equations (4.5) are valid in this case also. However, the Darcy law (3.5) should be replaced by

$$q_1 = -\chi_1 \partial p/\partial x_1, \quad q_2 = 0, \quad q_3 = -\chi_3 \partial p/\partial x_3, \tag{5.1}$$

where **q** is the fluid flux and χ_i the Darcy conductivity in the x_i direction. Consequently, the fluid diffusion equation (4.4) is replaced by the equation (Singh *et al.*, 2007)

$$\frac{\partial}{\partial t}\left[\sigma_{11} + \sigma_{33} + \frac{\alpha_0}{\nu_u - \nu} p\right] = \frac{\alpha_0(1-\nu_u)}{(1-\nu)(\nu_u - \nu)} \nabla_{13}^2 p \tag{5.2}$$

where

$$\nabla_{13}^2 = c_1 \frac{\partial^2}{\partial x_1^2} + c_3 \frac{\partial^2}{\partial x_3^2} \tag{5.3}$$

and

$$c_i = \frac{2G(1-\nu)(\nu_u - \nu)}{\alpha_0^2(1-\nu_u)} \chi_i \quad (i = 1, 3) \tag{5.4}$$

is the hydraulic diffusivity.

The four unknowns σ_{11}, σ_{33}, σ_{13} and p are to be determined from the coupled system of the four equations listed in (4.3), (4.5) and (5.2). This system can be solved in terms of Biot's stress function F introduced by equation (4.7). Equations (4.5), (4.7) and (5.2) yield

$$\nabla^2\left(\nabla^2 F + 2\eta p\right) = 0, \tag{5.5}$$

$$\frac{\partial}{\partial t}\left[\nabla^2 F + \frac{\alpha_0}{\nu_u - \nu} p\right] = \frac{\alpha_0(1-\nu_u)}{(1-\nu)(\nu_u - \nu)} \nabla_{13}^2 p \tag{5.6}$$

Eliminating F and p in turn, equations (5.5) and (5.6) lead us to the following decoupled equations

$$\left(\nabla_{13}^2 - \frac{\partial}{\partial t}\right)\nabla^2 p = 0, \tag{5.7}$$

$$\left(\nabla_{13}^2 - \frac{\partial}{\partial t}\right)\nabla^4 F = 0 \tag{5.8}$$

Taking the Laplace transform of equations (5.7) and (5.8) with respect to time and then solving the resulting equations in space variables we obtain the general solution of these equations. For the consolidation of a homogeneous poroelastic half-space $z \geq 0$ by surface loads, suitable solutions are of the form

$$p=\int_0^\infty\left(A_1e^{-mz}+A_2e^{-kz}\right)\begin{pmatrix}\sin & kx\\ \cos & kx\end{pmatrix}dk, \tag{5.9}$$

$$F=\int_0^\infty\left[B_1e^{-mz}+\left(B_2+B_3kz\right)e^{-kz}\right]\begin{pmatrix}\sin & kx\\ \cos & kx\end{pmatrix}dk, \tag{5.10}$$

where the arbitrary constants A_1, A_2, etc. may be functions of k, $x=x_1$, $z=x_3$ and

$$m=\left(\frac{c_1}{c_3}k^2+\frac{s}{c_3}\right)^{1/2}. \tag{5.11}$$

Inserting the expressions for F and p from equations (5.9) and (5.10) in equation (5.6) and (5.7), we find

$$A_1=\left[\left(k^2-m^2\right)/2\eta\right]B_1. \tag{5.12}$$

$$B_3=\frac{\alpha_0A_2}{2k^2(\nu_u-\nu)}\left[1+\frac{(c_1-c_3)(1-\nu_u)k^2}{(1-\nu)s}\right]. \tag{5.13}$$

Equations (4.7), (5.1), (5.9) and (5.10) yield

$$q_1=-\chi_1\int_0^\infty\left(A_1e^{-mz}+A_2e^{-kz}\right)\begin{pmatrix}\cos & kx\\ -\sin & kx\end{pmatrix}kdk, \tag{5.14}$$

$$q_3=\chi_3\int_0^\infty\left(mA_1e^{-mz}+kA_2e^{-kz}\right)\begin{pmatrix}\sin & kx\\ \cos & kx\end{pmatrix}dk, \tag{5.15}$$

$$\sigma_{11}=\int_0^\infty\left[B_1m^2e^{-mz}+\left\{B_2+B_3\left(kz-2\right)\right\}k^2e^{-kz}\right]\begin{pmatrix}\sin & kx\\ \cos & kx\end{pmatrix}dk, \tag{5.16}$$

$$\sigma_{33}=-\int_0^\infty\left[B_1e^{-mz}+\left\{B_2+B_3kz\right\}e^{-kz}\right]\begin{pmatrix}\sin & kx\\ \cos & kx\end{pmatrix}k^2dk, \tag{5.17}$$

$$\sigma_{13}=\int_0^\infty\left[B_1me^{-mz}+\left\{B_2+B_3\left(kz-1\right)\right\}ke^{-kz}\right]\begin{pmatrix}\cos & kx\\ -\sin & kx\end{pmatrix}kdk. \tag{5.18}$$

Corresponding to the stresses given by equations (5.16) to (5.18), the displacements are found by expressing strains in terms of displacements in the constitutive equations (4.2) and then integrating the resulting equations. We find ,

$$2Gu_1 = -\int_0^\infty \left[B_1 e^{-mz} + \left\{ B_2 + B_3 \left(2\nu - 2 + kz\right) + \frac{\alpha_0}{k^2} A_2 \right\} e^{-kz} \right] \begin{pmatrix} \cos & kx \\ -\sin & kx \end{pmatrix} kdk, \tag{5.19}$$

$$2Gu_3 = \int_0^\infty \left[B_1 m e^{-mz} + \left\{ B_2 + B_3 \left(1 - 2\nu + kz\right) - \frac{\alpha_0}{k^2} A_2 \right\} k e^{-kz} \right] \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} dk \tag{5.20}$$

Equations (5.9) and (5.14) to (5.20) constitute an analytical solution of the governing equations representing the diffusion-deformation of a poroelastic half-space possessing anisotropic permeability and compressible fluid and solid constituents. The three arbitrary constants B_1, B_2 and B_3 are to be determined from the boundary conditions. This solution has been used by Singh et al. (2007) to study the quasi-static deformation of a half-space by surface loading. The problem of the consolidation of a uniform half-space caused by normal disc loading has been discussed in detail. The following conclusions were drawn in relation to the consolidation of a poroelastic half-space by normal loads:

(i) The anisotropy in permeability may accelerate the consolidation process. However, it has no effect on the initial and the final settlements.

(ii) The compressibility of the solid constituents of the poroelastic medium may accelerate the consolidation process. However, it has no influence on the initial and final settlements.

(iii) In the short term, the compressibility of the solid constituents increases the pore pressure.

(iv) The compressibility of the fluid constituents increases the initial settlement. It has no influence on the final settlement.

(v) The compressibility of the fluid constituents decreases the short-term pore pressure.

Chen (2004) used the state vector method to investigate the consolidation of a multilayered poroelastic half-space with anisotropic permeability. However, he assumed the solid constituent of the poroelastic material to be incompressible. For such a poroelastic material, the Biot-Willis co-efficient $\alpha = 1$. Therefore, only three constitutive poroelastic constants are involved in his formulation as against four constitutive constants used here which define a poroelastic material with compressible solid and fluid constituents.

We have considered the above plane strain consolidation problem. The case of axial symmetry can be tackled by a similar procedure (Singh *et al.*, 2009). Further, equations (5.9) and (5.10) contain only negative exponentials suitable for a half-space. For a clay layer, we should choose solutions for p and F containing both negative and positive exponentials (Rani *et al.*, 2011).

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