DOI: 10.15415/mjis.2014.22016

Fractals Generated by Various Iterative Procedures – A Survey

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Abstract These days fractals and the study of their dynamics is one of the emerging and interesting area for mathematicians. New fractals for various equations have been created using one-step iterative procedure, two-step iterative procedure, three-step iterative procedure and four-step iterative procedure in the literature. Fractals are geometric shapes that have symmetry of scale. In this paper, a detailed survey of fractals existing in the literature such as Julia sets, Mandelbrot sets, Cantor sets etc have been given.

Keywords: Julia Sets, Mandelbrot Sets, Cantor Sets, Sierpinski's Triangle, Koch Curve

1. INTRODUCTION

In the past, mathematics has been concerned largely with sets and functions to which the methods of classical calculus be applied. Sets or functions that are not sufficiently smooth or regular have tended to be ignored as 'pathological' and not worthy of study. Certainly, they were regarded as individual curiosities and only rarely were thought of as a class to which a general theory might be applicable. In recent years this attitude has changed. It has been realized that a great deal can be said and is worth saying, about the mathematics of non-smooth objects. Moreover, irregular sets provide a much better representation of many natural phenomena than do the figures of classical geometry. Fractal geometry provides a general framework for the study of such irregular sets (Falconer (1990)).

Generally, people believe that the geometry of nature is centered on the simple figures like lines, circles, polygons, spheres, and quadratic surfaces and so on. But there are so many examples in nature which shows that the geometry does not depend on simple figures. Can we describe the structures of animals and plants? What is the shape of mountain? Like these structures there are many objects in nature which are complicated and irregular. Moreover, dynamical behavior in nature also can be complicated and irregular. What is the mathematics behind heart and brain waves as seen in electro-diagrams,

Mathematical Journal of Interdisciplinary Sciences
Vol. 2, No. 2,
March 2014
pp. 195–208

especially when a sudden fibrillation occurs that might cause failure? What is the mathematical model of ups and downs in financial market or even social behavior? How do we model turbulent weather and cascading waterfall? (Rani (2002)).

To analyze many of these questions, fractal and mathematical chaos are appropriate tools. Fractals and mathematical chaos are the frontiers of science and play significant roles in the study of science, medicine, business, textile industries and also in the other areas. There are several books which give basic ideas in fractals and chaos theory. Book of Gullick (1992) include an account of Chaos in one-dimension. The first book by Mandelbrot (1982) in the theory of fractals includes different fractal shapes in the nature. The books by Falconer (1990), Edgar (2008), Bransely (1993) and Tricot (1995) consist of basic results in the theory of fractal and dimension. Beardon (1991), Pietgen et al. (1986, 1992b) and Devaney (1992) give a nice account of the results in complex dynamics.

The purpose of this paper is to give a detailed study of new fractals created by one step iterative procedure, two-step iterative procedure, three-step iterative procedure and four-step iterative procedure.

2. FRACTALS

Generally, fractals apply to static geometric objects e.g. plants, networks of veins, and freeze frame images of a waterfall etc. The word 'Fractal' was coined by Benoit Mandelbrot (see Bunde (1992, p-2)) in his fundamental essay from the Latin word fractus, meaning broken, to describe objects that were too irregular to fit into a traditional geometric setting (cf. Peitgen et al. (2004)). Indeed, Mandelbrot introduced the term fractal in 1975 and defined a fractal as a set whose Hausdorff dimension (fractal dimension) is strictly greater than its topological dimension (Crownover (1995, p. 109)). The name was actually given because fractals exist in fractional dimensions. To visualize a fractal, consider a head of cauliflower or a bunch of broccoli. If a piece is broken off from either of these, the part still resembles the whole. Sometimes the resemblance may be weaker than strict geometrical similarity; for example similarity may be approximate or statistical.

When we refer to a set as a fractal, therefore they will typically have the following in mind:

(i) It has a fine structure at arbitrarily small scales.
(ii) It is too irregular to be described in traditional geometrical language, both locally and globally.

(iii) Often it has some form of self-similarity, perhaps approximate or statistical.
(iv) Usually the fractal dimension of fractal is greater than its topological dimension.
(v) In most cases of interest fractal is defined in a very simple way, perhaps recursively.

A mathematical fractal is based on an equation that undergoes iteration, a form of feedback based on recursion. Following are the results of sets that are commonly referred to as fractals.

2.1 Julia Sets

With the introduction of fractal geometry, mathematics has presented some interesting complex objects to computer graphics. Interest in Julia sets and related mathematics began in 1920's with Gaston Julia (Pietgen et al. (2004), p. 122)). What makes Julia sets interesting to study is that despite being born out of apparently simple iterative processes they can be very intricate and often fractal in nature (Pietgen et al. (2004), p. 122)). Now, fractal theory is incomplete without the presence of Julia sets. Julia sets have been studied for quadratic (Crilly et al. (1991), Devaney (1992), Lei (1990), Pietgen et al. (2004)), cubic (Branner (1988, 1992), Devaney (1992), Doudy (1984), Epstein (1999), Frame (1992), Liaw (1998)) and higher degree polynomials (Geum (2009)), under Picard orbit, which is an example of one-step feedback process. In the last few decades many beautiful Julia sets and their applications in various branches of science such as mathematics, engineering, computer science, medical science, etc have been studied using two-step feedback process (superior orbit) and three-step feedback process (I- superior orbit) (for instance see Chauhan et al. (2010a, b)). Following is the definition of Julia sets for $Qc(z) = z^n + c$, where n = 2, 3 .....

Definition 1. The set of points *K* whose orbits are bounded under the function iteration of $Q(z)$ is called the filled Julia set. Julia set of Q is the boundary of the filled Julia set *K*. The boundary of a set is the collection of points for which every neighborhood contains an element of the set as well as an element, which is not in the set (see Crownover (1995) and Devaney (1992)).

The following theorem gives the general escape criterion of Julia sets and its corollaries further refine the escape criterion for computational purposes using Picard orbit (see Beradon (1991), Crownover (1995), Devaney (1992), Pietgen et al. (1992a)).

Theorem 1. Suppose $|z| \geq |c| > 2$, where c is in the complex plane. Then we have $\left|G_c^{\,n}(z)\right| \to \infty$ as $n \to \infty$

Corollary 1. Suppose $|c| > 2$. Then the orbit of 0 escapes to infinity under Q_c.

Corollary 2. Suppose $|z| > \max\{|c|, 2\}$. Then $\left|G_c^{\,n}(z)\right| > \left|(1+\lambda)^n\right| |z|$ and SO $\left|G_c^{\,n}(z)\right| \to \infty$ as $n \to \infty$, where λ is a positive number.

Corollary 3. Suppose for some K ≥ 0, we have $\left|G_c^{\,k}(z)\right| > \max\left\{|c|, 2\right\}$. Then $\left|G_c^{\,k+1}(z)\right| > (1+\lambda)\left|G_c^{\,k}(z)\right|$, SO $\left|G_c^{\,n}(z)\right| \to \infty$ as $n \to \infty$.

In 2004, Rani introduced superior iterates (a two-step feedback process) in the study of fractal theory, jointly with Kumar, and created superior Julia sets (Rani et al. (2004)). The following theorem gives the general superior escape criterion for Julia sets and its corollaries further presents the escape criterion for computational purposes using Superior orbit.

Theorem 2. (General escape criterion) For general function ,$G_c(z) = z^n + c$ n = 1, 2, 3, …, where $0 < \alpha \leq 1$ and c is in the complex plane. Define

$$z_1 = (1-\alpha)z + \alpha G_c(z),$$

$$.....,$$

$$z_n = (1-\alpha)z_{n-1} + \alpha G_c(z_{n-1})$$

for n = 2, 3, ... Thus, the general escape criterion is $\left\{|c|, (2/\alpha)^{1/n-1}\right\}$.

Corollary 4. Suppose that $|c| > (2/\alpha)^{1/n-1}$. Then, the superior orbit $SO(G_c, 0, \alpha_n)$ escapes to infinity.

Corollary 5. Suppose for some $k \geq 0$, we have $|z_k| > \max\left\{|c|, (2/\alpha)^{1/k-1}\right\}$. Then $|z_{k+1}| > (1+\lambda)\,|z_k|$; so $|z_n| \to \infty$ as $n \to \infty$.

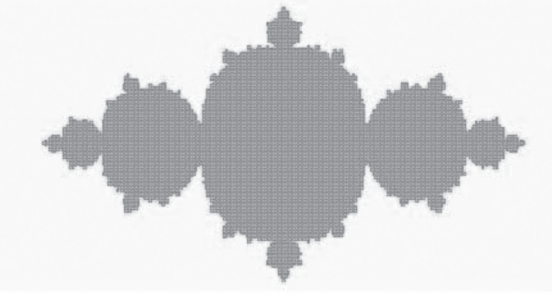

Figure 1: Superior Julia Sets for Quadratic Map.

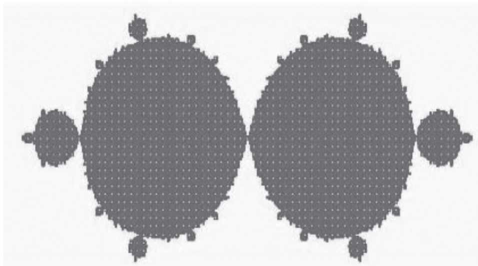

Figure 2: Superior Julia Sets for Cubic Map.

This corollary gives a general algorithm for computing filled superior Julia sets for the function of the form $G_c(z) = z^n + c$, n= 1, 2, 3, … .

Fig 1 and Fig 2, represents the superior Julia sets for quadratic and cubic maps using two step iterative procedures.

Recently, Chauhan et. al. (2010a, b) generated new Julia sets via Ishikawa iterates (an example of three-step feedback process). The following theorem gives the general I-superior escape criterion for Julia sets and its corollaries further presents the escape criterion for computational purposes using I-Superior orbit.

Theorem 3. For general function $G_c(z) = z^n + c$, n= 1, 2, 3, … , where $0 < \alpha \leq 1$, $0 < \beta \leq 1$ and c is in the complex plane. Define

$$z_1 = (1-\alpha)z + \alpha G_c(z),$$
$$.....,$$
$$z_n = (1-\alpha)z_{n-1} + \alpha G_c(z_{n-1})$$

Thus, the general escape criterion is $\{|c|, (2/\alpha)^{1/n-1}, (2/\beta)^{1/n-1}\}$.

Corollary 6. Suppose that $|c| > (2/\alpha)^{1/n-1}$ and $|c| > (2/\beta)^{1/n-1}$ exists. Then, the I-Superior orbit *ISO* $(G_c, 0, \alpha_n, \beta_n)$ escapes to infinity.

Corollary 7. Suppose for some $k \geq 0$, we have $|z_k| > \max\{|c|, (2/\alpha)^{1/k-1}, (2/\beta)^{1/k-1}\}$. Then $|z_{k+1}| > \gamma|z_k|$, *so* $|z_n| \to \infty$ *as* $n \to \infty$ .This corollary gives a general algorithm for computing I- superior Julia sets for the function of the form $G_c(z) = z^n + c$, $n = 1, 2, 3, \ldots$.

Fig 3 and Fig 4, represents I- Superior Julia sets for the quadratic and cubic maps.

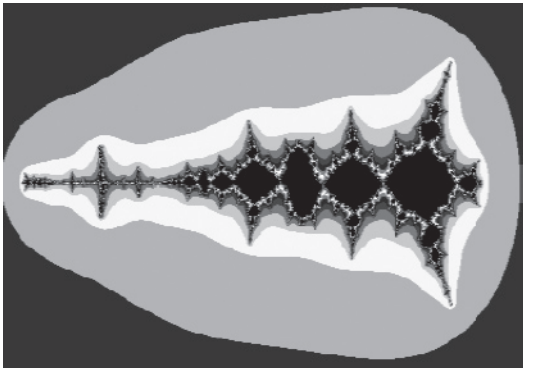

Figure 3: I-Superior Quadratic Julia Sets.

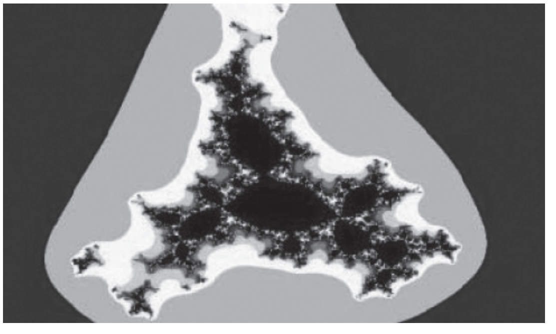

Figure 4: I-Superior Cubic Julia Sets.

2.2. Mandelbrot Sets

In 1975, Benoit Mandelbrot extended the work of Gaston Julia and introduced the Mandelbrot set; a set of all connected Julia sets. Mandelbrot sets have been studied for quadratic (Crilly et al. (1991), Devaney (1992), Lei (1990), Pietgen et al. (2004)), cubic (Branner (1988, 1992), Devaney (1992), Doudy (1984), Epstein (1999), Frame (1992), Liaw (1998)) and higher degree polynomials (Geum (2009)), under Picard orbit, which is an example of one-step feedback process. Every Julia set for a function is either connected or disconnected. The Mandelbrot set works as a locator for the two types of Julia sets. Each point in the Mandelbrot set shown in Figure 5 represents a c-value for which the Julia set is connected and each point in its complement represents a c-value for which the Julia set is disconnected (cf. Crownover (1995), Devaney (1992), Gutfraind (1990), Pietgen et al. (1992a, b)). Following is the definition of Mandelbrot set for Qc(z) = z^n+c where n = 2, 3, ...

Definition 3. (Mandelbrot Set). The Mandelbrot set *M* consists of all parameters *c* for which the filled Julia set of Q_c is connected, that is.

$$M = \{c \in C : K(Q_c) \text{ is connected}\}.$$

In fact, *M* contains an enormous amount of information about the structure of Julia sets. The Superior Mandelbrot set *SM* for the Quadratic $Q_c(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is

$$SM = \{c \in C : \{Q_c^n(0)\}; n = 0,1,2,.... \text{is bounded}\}.$$

we choose the initial point 0, as 0 is the only critical point of Q_c (Pietgen et al. (2004), p. 249).

Similarly, the Mandelbrot set *ISM* for Qc(z) = $z^n + c$, where n = 2, 3, . . . with respect to Ishikawa iterates is called I-superior Mandelbrot set.

Escape criterions play a crucial role in the analysis and generation of Mandelbrot, superior Mandelbrot set (Rani et al. (2004)) and I-superior Mandelbrot set (Chauhan et al. (2010)). The escape criterions studied above in the section of Julia set are applicable in the generation of superior Mandelbrot sets and I-superior Mandelbrot sets. Following are the general escape criterions of Mandelbrot set with some attractive figures::

- General escape criterion of Mandelbrot set for Qc(z) = $z^n + c$, is $|z_k| > \max\{|c|, 2\}$. Fig 5, shows the Mandelbrot sets generated by Picard Orbit.

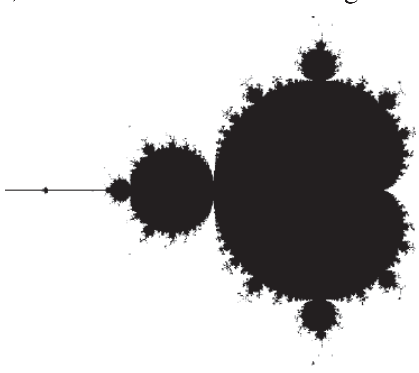

Figure 5: Mandelbrot set.

- General escape criterion of superior Mandelbrot set for $Qc(z) = z^n + c$, is $|z_k| > \max\{|c|, (2/\alpha)^{1/k-1}\}$. Fig 6 and Fig 7, shows the Mandelbrot sets for quadratic and cubic maps generated by two step iterative procedure.

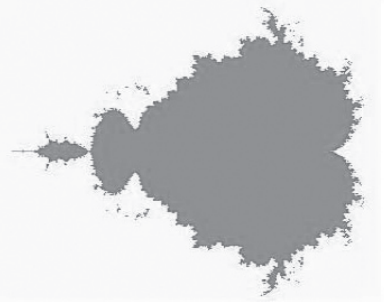

Figure 6: Superior Quadratic Mandelbrot sets.

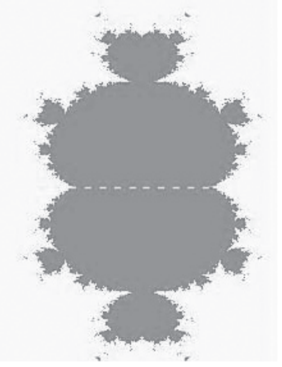

Figure 7: Superior Cubic Mandelbrot sets.

- General escape criterion of I-superior Mandelbrot set for $Qc(z) = z^n + c$, is $|z_k| > \max\{|c|, (2/\alpha)^{1/k-1,}(2/\beta)^{1/k-1}\}$. Fig 8 and Fig 9, shows the Mandelbrot sets for quadratic and cubic maps generated by three step iterative procedure.

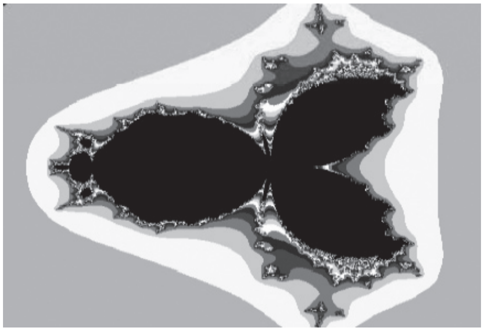

Figure 8: I-Superior Quadratic Mandelbrot sets.

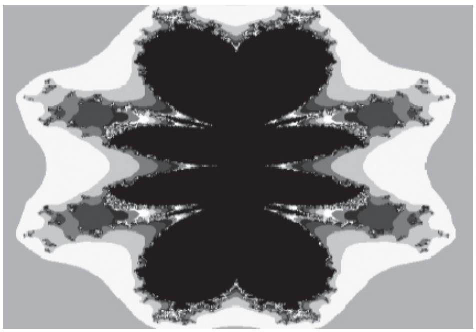

Figure 9: I-Superior Cubic Mandelbrot sets.

In order to generate the Mandelbrot set graphically, the computer screen becomes the complex plane. Each point on the plane is tested into the equation $z = z^2 + c$. If the iterated z stayed within a given boundary forever, i.e. converges, the point is inside the set and we have plotted the point in magenta. If the iteration diverges, the point was plotted in a color with respect to how quickly it escaped. For a detailed analysis of the Mandelbrot set, one may refer to Crilly (1991), Mann (1953), Mandelbrot et al. (1982, 1991, 1996, 1998), Peitgen et al. (1986, 1988, 1992a, 1992b, 2004) and Reeve (1991)..

The implementation difference between the Julia set and the Mandelbrot set is the way in which the function is iterated. The Mandelbrot set iterates $z = z^2 + c$ with z always starting at 0 and varying the c value. The Julia set iterates $z = z^2 + c$ for a fixed c value and varying z values. In other words, the Mandelbrot set is in the parameter space, or the c-plane, while the Julia set is in the dynamical space, or the z-plane. For a detailed study, one may refer Crownover (1995), and Milnor (1990).

Figure 10: George Cantor Set

2.3. Cantor Set

The Cantor set is a classical example of Fractal theory. By a general Cantor set, we mean a set of points lying on a single line segment that has a number of remarkable and deep properties (Cantor (1883)). It was discovered in 1875 by Smith (1875) and first introduced by German mathematician George Cantor (1845 - 1918) that became known as Cantor ternary set (Cantor (1883)). The Cantor set finds a celebrated place in mathematical analysis and its applications. For some basic study on Cantor set one may refer to Beardon (1991), Devaney (1992), Falconer (1990) and Peitgen et al. (2004). The Cantor set has many interesting properties and consequences in the field of set theory, topology, and fractal theory (Bulaev (2000), Fleron (1994), Gutfrained (1990), Horiguchi (1984a, b), Tsuji (1953)). Also, for more applications of Cantor set in discrete dynamical system and mathematical analysis, one may refer to (Ferienos (1999), Lee (1998), Rahman).

Recently, Rani (2011), introduced the Superior Cantor sets and presented them graphically by Devil's staircases. They generated new Cantor sets by two methods. In one method, initiator is divided into three equal parts and either left segment or right segment of initiator is dropped. In another method, unequal division of initiator has been done. The interesting point here is that some of the Cantor sets given by Rani et al. (2010) are common to Cantor sets given by Shaver (2010).

Figure 11: Sierpinski gasket

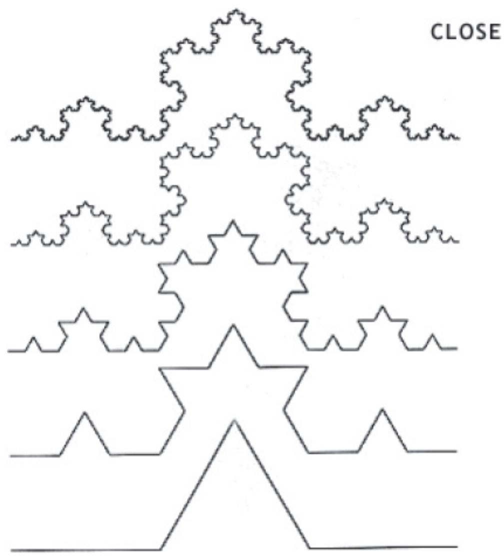

Figure 12: Koch Curve

Definition 2. (Cantor set) The Cantor C is defined as $C = \cap_{n=1}^{\infty} I_n$, where I_{n+1} is constructed by trisecting I_n and removing the middle third, I_0 being the closed interval $0 \leq x \leq 1$ (Branner and Hubbard (1988)). Fig 10, shows the Cantor middle one third set due to George Cantor (1992).

Besides the sectors mentioned above, chaos and fractals are the new frontiers of science and have come to play significant roles in the study of applicable areas of sciences, medicine, business, textile industries, music and several other areas of human activity (see, for instance, Barnsley (1988), Beardon (1991), Horn (1991), Rani et al. (2010)).

2.4. Sierpinski Triangle(Gasket)

Sierpinski's Triangle (or gasket), was introduced by the great Polish mathematician Waclaw Sierpinski (1882-1969) in 1916. He was one of the most influential mathematicians of his time in Poland and had a worldwide reputation. He described some of its interesting properties in 1916 (Pietgen et al. (2004)).

The basic geometric construction of the Sierpinski triangle goes as follows. We begin with a triangle in the plane and then apply a repetitive scheme of operations to it. Pick the midpoints of its three sides. Together with the old vertices of the original triangle, these midpoints define four congruent triangles of which we drop the center one. This completes the basic construction step.

In other words, after the first step we have three congruent triangles whose sides have exactly half the size of the original triangle and which touches at three points which are common vertices of two contiguous triangles. Now, we follow the same procedure with the three remaining triangles and repeat the basic step as often as desired (Devaney (1992), Pietgen et al. (2004)).

2.5. Koch Snowflake and Koch curve :

Helge von Koch was a Swedish mathematician who, in 1904, introduced what is now known as the Koch Curve. The basic geometric construction of the Sierpinski triangle goes as follows. Begin with a straight line. This initial object is also called initiator. Partition it into three equal parts. Then replace the middle third by an equilateral triangle and take away its base. This completes the basic construction step. A reduction of this figure, made of four parts will reused in the following steps. Thus, we now repeat, taking each of the resulting line segments, partitioning them into three equal parts, and so on. Self similarity is built into the construction process, i. e. each part of the 5 parts in the k^{th} step is again a scaled down version by a factor of 3 of the entire curve in the previous $(k\text{-}1)^{th}$ step (Devaney (1992), Pietgen et al. (2004)). Figure 12, shows the Koch Curve created by Helge von Koch in 1904.

3. CONCLUSION

The Julia sets, Mandelbrot sets, Cantor sets, Sierpinski's Triangle (or gasket) and Koch Curve all are examples of fractal sets. In this paper, a survey of fractals for various equations created by using one-step iterative procedure, two-step iterative procedure, three-step iterative procedure and four step iterative procedure have been given. The following survey has been drawn:

1. A detail survey of Julia sets and Mandelbrot sets with some beautiful pictures have been given in subsection 2.1 and 2.2 respectively.
2. Further in subsection 2.3, results on Cantor sets have been presented which is a classical example of fractal in literature.
3. In subsection 2.4 and 2.5, the Sierpinski's Triangle (or gasket) and Koch Curve have been presented.

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