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Finite Groups with Two Class Sizes of Some Elements

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Abstract Let G be a finite group. We prove that if $\{1,m\}$ are the conjugacy class sizes of p-regular elements of primary and biprimary orders of G, for some prime p, then G has Abelian p-complement or $G = PQ \times A$, with $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ and $A \subseteq Z(G)$, with q a prime distinct from p. As a consequence, if $\{1, m\}$ are the conjugacy class sizes of ab p-regular elements of primary and biprimary orders of G, then $m = p^a q^b$. In particular, if $b = 0$ then G has abelian p-complement and if $a = 0$ then $G = P \times Q \times A$ with $A \subseteq Z(G)$.

Keywords: conjugacy class sizes, nilpotent groups, finite groups. MSC: 20D10; 20D20

1. INTRODUCTION

All groups considered in this paper are finite. If G is a group, then x^G denotes the conjugacy class containing x, $|x^G|$ the size of x^G (following Baer, (1953)) we call $\mathrm{Ind}_G(x) = |x^G| = |G : C_G(x)|$, the index of x in G). The rest of our notation and terminology are standard. The reader may refer to Robinson (1983).

It is well known that there is a strong relation between the structure of a group and the sizes of its conjugacy classes and there exist several results studying the structure of a group under some arithmetical conditions on its conjugacy class sizes. For example, Itô (1953) shows that if the sizes of the conjugacy classes of a group G are $\{1, m\}$, then G is nilpotent, $m = p^a$ for some prime p and $G = P \times A$, with P a Sylow p-subgroup of G and $A \subseteq Z(G)$. In Beltrán and Felipe (2003) proved a generalization of this result for p-regular conjugacy class sizes and some prime p, with the assumption that the group G is p-solvable. Recently, in Alemany et al. (2009), they improved this result by showing that the p-solvability condition is not necessary. In the present paper, we improve this result by replacing conditions for all p-regular conjugacy classes by conditions referring to only some p-regular conjugacy classes.

Theorem A *Let G be a finite group. If $\{1, m\}$ are the conjugacy class sizes of p-regular elements of primary and biprimary orders of G, for some prime*

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CHITKARA UNIVERSITY

p, then G has Abelian p-complement or $G = PQ \times A$, *with* $P \in Syl_p(G)$, $Q \in Syl_q(G)$ and $A \subseteq Z(G)$, *with q a prime distinct from p. As a consequence, if* $\{1, m\}$ *are the conjugacy class sizes of p-regular elements of primary and biprimary orders of G, then* $m = p^a q^b$. *In particular, if* $b = 0$ *then G has abelian p-complement and if* $a = 0$ *then* $G = P \times Q \times A$ *with* $A \subseteq Z(G)$.

2. PROOF OF THEOREM A

In order to prove our main result, we need the following two important lemmas.

Lemma 2.1 *Let G be a group. Then the following two conditions are equivalent:*

(i) 1 and $m > 1$ *are the only lengths of conjugacy classes of* p' *-elements of primary and biprimary orders of G;*

(ii) 1 and $m > 1$ *are the only lengths of conjugacy classes of* p' *-elements of G.*

PROOF (I)⇒(II)

Let a be any q-element of index m and b be any r-element of $C_G(a)$, where q $q \neq p$ and $r \neq p$. Notice that and since m is the largest conjugacy class size of p' -elements of primary and biprimary orders of G, then $C_G(ab) = C_G(a)$ and hence $C_G(a) \subseteq C_G(b)$. This implies that $b \in Z(C_G(a))$.

$$C_G(ab) = C_G(a) \cap C_G(b) \subseteq C_G(a)$$

Now let x be any non-central p'-element of G and write $x = x_1 x_2 \cdots x_s$, $s \geq 3$, where the order of each x_i is a power of a prime p_i ($p_i \neq p$, $i = 1, 2, \ldots, s$) and the x_i commute pairwise. As x is a non-central p' -element of G, we know that at least one of the x_i such that x_i is non-central. Without loss of generality, we can assume that x_1 is non-central. Now

$$\begin{aligned} C_G(x) &= C_G(x_1 x_2 \ldots . x_s) \\ &= C_G(x_1) \cap C_G(x_2 \ldots . x_s) \\ &= C_G(x_1) \cap C_G(x_2) \cap \ldots . \cap C_G(x_s) \\ &\subseteq C_G(x_1), \end{aligned}$$

and by the previous argument we may conclude that have that $x_i \in Z(C_G(x_1))$ for $i = 2, \cdots, s$. Hence we get that $C_G(x_1) \leq C_G(x_i)$, $i = 2, \cdots, s$. Thus

$$\begin{aligned} C_G(x) &= C_G(x_1 x_2 \ldots . x_s) \\ &= C_G(x_1) \cap C_G(x_2 \ldots . x_s) \\ &= C_G(x_1) \cap C_G(x_2) \cap \ldots . \cap C_G(x_s) \\ &= C_G(x_1) \end{aligned}$$

It follows that the conjugacy class size of x is equal to the conjugacy class size of x_1, that is, m.

Lemma 2.2[5.Theorem A] *Let G be a finite group. If the set of p-regular conjugacy class sizes of G has exactly two elements, for some prime p, then G has Abelian p-complement or* $G = PQ \times A$, *with* $P \in Syl_p(G)$, $Q \in Syl_q(G)$ *and* $A \subseteq Z(G)$, *with q a prime distinct from p. As a consequence, if {1, m} are the p-regular conjugacy class sizes of G, then m = paqb . In particular, if b = 0 then G has abelian p-complement and if a = 0 then* $G = P \times Q \times A$ *with* $A \subseteq Z(G)$.

Proof of Theorem A By Lemma 2.1 and 2.2, Theorem A holds.

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