DOI: 10.15415/mjis.2014.22013

A New Attempt to Construct the Laplace Operator on Fractals

Krzysztof Drachal*

Faculty of Mathematics and Information Science at the Warsaw University of Technology ul. Koszykowa 75, 00 – 662 Warszawa, Poland

*Email: k.drachal@mini.pw.edu.pl

Abstract One of the most important topics in the analysis on fractals is to construct the Laplacian. But this is actually a particular case of a wider problem – to construct geometrical objects on fractals. Currently studied methods sometimes lead to difficult problems, require wide knowledge from different branches of mathematics or does not lead to any strict computational methods, which could be easily applied for example in engineering.

In this paper a new attempt is presented. Fractals are treated as objects from so called differential spaces, i.e. broader category than manifolds. The usefulness of differential spaces is shown in particular fractal situations, when one studies some „weird" subsets of $\mathbb{R}^n$, which are not manifolds themselves.

Keywords: fractals, Laplace operator on fractals, laplacian on fractals, analysisonfractals, geometricobjectsonfractals, differential spaces.

1. INTRODUCTION

Recently the analysis on fractals evolved rapidly. This branch of mathematics focuses on generalisation of calculus on smooth manifold to calculus on fractals. One of the most important topics in this branch of mathematics is to construct the Laplacian. It is actually a particular case of a wider problem – to construct differential operators on fractals. And this is done in various ways, mainly on probabilistic, analytical or measure theoretic background (Barlow, 1998; Hambly and Kumagai, 1999; Jonsson and Wallin, 1984; Harrison, 1998; Mosoc, 1998; Strichartz, 2006). Surprisingly no geometrical attempt was considered yet.

For example Kigami approximates a fractal from within by a sequence of finite graphs and obtains the Laplacian as the renormalised limit of graph Laplacians (Kigami, 1989, 2001). The attempt proposed in Jonsson and Wallin (1984) is a bit similar to the one proposed in the below paper. It bases on restricting function from $\mathbb{R}^n$ to a fractal. In Barlow (1998) a probabilistic attempt is widely discussed. The main advantage of such an attempt is that e.g. heat equation is well estimated. On the other hand studying this equation

Mathematical Journal of Interdisciplinary Sciences
Vol. 2, No. 2,
March 2014
pp. 171–177

by methods proposed in Kigami (2001) lead to serious problems. However the method in Barlow (1998) requires wide knowledge from probability theory and does not lead to any strict computational methods (able to easily applied e.g. in engineering).

In this paper a new attempt is presented. Fractals are treated as objects living not on manifolds, but on so called differential spaces. Differential spaces are broader category than manifolds (Sikorski, 1967). However whole differential geometry may be easily developed on them. The usefulness of differential spaces emerges for example in situations when one studies some „weird" subsets of $\mathbb{R}^n$, which are not manifolds themselves. In this paper tangent vectors, differential forms, vector fields, etc. are constructed on fractals.

It is interesting that differential spaces were heavily studied in order to apply to cosmological problems. For example classically a singularity is not treated as a part of a space–time itself, but it may be included into „differential space–time". Therefore e.g. the behaviour of such generalised differential forms and metrices was thoroughly studied, but it is surprising that little study was done to apply this generalisation to fractals. Also in studying Laplacian constructions on fractals no geometric method was discussed yet.

2. BASICS OF DIFFERENTIAL SPACES

For sake of clarity only the necessary facts about differential spaces will be given here. In Buchner et al. (1993) one may find a list of papers about differential spaces with brief description about the content.

Suppose one is given a set M. Let denote by C_0 a set of some real functions on M, i.e. $C_0 := \{f_1,...,f_n\}$, where $f_i : M \to \mathbb{R}$ for all $i = 1,...,n$. Now one may consider a topological space (M, τ), where τ is the weakest topology on M in which all functions from C_0 are continuous.

Def. 2.1. *Function f is called a local C_0–function, if for every point $x \in M$ there is a neighbourhood $U \in \tau$ and $g \in C_0$, such that $f|U = g|U$.*

The set of all local C_0–functions on M is denoted by $(C_0)_M$.

Def. 2.2. $sc(C_0) := \{\omega \circ (f_1,...,f_n) \mid \omega \in C^\infty(\mathbb{R}^n), f_1,...,f_n \in C_0\}$

The above family of functions is called superposition closure.

Def. 2.3. *C is called a differential structure on M, if it is closed with respect to localisation ($C = C_M$) and closed with respect to superposition with smooth Euclidean functions ($C = scC$).*

Def. 2.4. *A pair (M, C) such that M is an arbitrary set, and C is a family of functions such that $C = (scC)_M$ is called a differential space.*

Def. 2.5. *If $C_0 := \{f_1,...,f_n\}$ is some family of real functions on M and $C = (scC_0)_M$ then the pair (M, C) would be called differential space generated by C_0. It is denoted by $C = genC_0$.*

For example $(\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ is generated by projections $C_0 = \{\pi_1,...,\pi_n\}$, where $\pi_i : \mathbb{R}^n \to \mathbb{R}^n$ for $i = 1,...,n$. It is a classical n–dimensional smooth manifold. However by considering more „unusual" functions in family C_0 one may obtain an object which would not be a smooth manifold.

Def. 2.6. *Mapping $F : M \to N$ is called smooth, if $f \circ F \in C$ for all $f \in D$.*

It can be simply proved that in order to verify smoothness one does not have to check all functions from D, but it is enough to check smoothness on generators from D_0.

Def. 2.7. *F is called diffeomorphism, if it is injective and both F and F^{-1} are smooth (in the above sense).*

In case there exist some fixed $n \in N$ and a countable (or at least finite) covering $\{A\}_{i\in I}$ of M such that for all $i \in I$ there exists diffeomorphism $F_i : (A_i, C_{Ai}) \to (\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ then (M, C) is a smooth manifold (in a classical sense). This definition is equivalent to the more common one using charts. However it is interesting that differential calculus and differential geometry can be studied also on differential spaces (as their names suggests so),which are not differential manifolds. Smooth manifold are only a special case of differential spaces. This is why differential spaces are generalisation of a classical manifold concept. Moreover one may notice that classically any function $f \in C^\infty(\mathbb{R}^n)$ is called smooth. Therefore if one considers a differential space (M, C), which might not be a manifold, the family of functions C can be treated as some analogue of family of classically smooth functions.

3. CONSTRUCTION OF GEOMETRIC OBJECTS

Def. 3.1. *If (M, C) is a differential space, then any linear mapping $v : C \to \mathbb{R}$, which satisfies the Leibnitz rule in $p \in M$, i.e. $v(fg) = v(f)g(p) + f(p)v(g)$, for all $f, g \in C$, is called a tangent vector to (M, C) at point p.*

All tangent vectors to (M, C) at p constitute a tangent vector space to (M, C) at p. This vector space is denoted by T_pM. Addition and multiplication by scalars are defined in the usual sense.

Def. 3.2. *A mapping* $X : M \to \bigcup_{p\in M}, T_pM\ p \to X(p) \in T_pM$ *is called a vector field on* (M, C).

Def. 3.3. *A tangent vector field X to* (M, C) *is called smooth, if for all* $f \in C$ *the function* $X(\cdot)(f)$: $M \to \mathbb{R}$, $X(\cdot)(f)$: $p \to X(p)(f)$ *belongs to C*.

Set of all smooth tangent vector fields to (M, C) is denoted by $X(M)$.

Def. 3.4. *If* $F :(M, C) \to (N, D)$ *is smooth, then* $dF : \bigcup_{p\in M} T_pM \to \bigcup_{q\in N} T_pN$, *defined by the formula* $(dF(v))(f) := v(f \circ F)$ *for all* $f \in D$, *is called a differential of mapping F*

In particular $dF\big|_{T_pM}$ is called a differential of F at point $p \in M$ and is denoted by dF_p.

Def. 3.5. *A triple* $((M, C), (TM, TC), \pi)$ *is called a tangent bundle to a differential space (M, C), if*:

- $TM := \bigcup_{p\in M} T_PM$,
- $\pi : TM \to M$ *is a projection*,
- $TC := gen(\{f \circ \pi \mid f \in C\} \cup \{df \mid f \in C\})$.

Consider $T^kM := \{(v_1,...,v_k) \in TM \times \cdots \times TM \mid \pi(v_1) = \cdots = \pi(v_k)\}$, $k \in N$, called Whitney sum. One has a product structure $TC \times \cdots \times TC$ on $TM \times \cdots \times TM$.

Def. 3.6. *Smooth mapping* $\omega : T^kM \to \mathbb{R}$ *is called a pointwise differential k–form, if for all* $p \in M$ $\omega_p := \omega|_{T_p^kM}$ *is* $\mathbb{R}$*–linear and skew–symmetric*.

The set of all pointwise differential k–forms on (M, C) is denoted by $A^k(M)$. They were studied by Kowalczyk (1980).

Def. 3.7. *A mapping* $\omega : \chi(M) \times \cdots \times \chi(M) \to C$ *is called a global differential form on* (M, C), *if this mapping is multilinear and skew–symmetric*.

The set of all global differential n–forms on (M, C) is denoted by $\Omega^n(M)$. For $\omega, \eta \in \Omega^n(M)$, $X_1,...,X_n \in \chi(M)$ and $f \in C$, if addition and multiplication is defined by the formulas $(\omega + \eta)(X_1,...,X_n) := \omega(X_1,...,X_n) + \eta(X_1,...,X_n)$ and $(f \cdot \omega)(X_1,...,X_n) := f\omega(X_1,...,X_n)$ then $\Omega^n(M)$ has a structure of C–module. $\Omega^0(M) := C$.

Def. 3.8. *The exterior product of k–form* ω *and l–form* η *is defined by the formula* $(\omega \wedge \eta)(X_1,...,X_k,X_{k+1}....X_{k+l}) := \sum_{\sigma\in S_{k+l}} \mathrm{sgn}(\sigma)\ \omega(X\sigma_{(1)},...,X\sigma_{(k)})\ \eta(X\sigma_{(k+1)},...,X\sigma_{(k+l)})$.

$\Omega(M) := \Omega(M) := \oplus_{n\geq 0}\Omega^n(M)$ is a graded algebra over $\mathbb{R}$.

Def. 3.9. *Exterior derivation, d, in $\Omega(M)$ is defined by the formula $(d\omega)(X) := X(\omega)$, if $\omega \in \Omega^0(M)$ and $X \in X(X)$. In case $\omega \in \Omega^k(M)$, $k \geq 1$ the formula is $(d\omega)(X_1,...,X_{k+l}) := \sum_{i=1}^{k+l} (-1)i{+}1\ X_i(\omega(X_1,..., \hat{X}_j,...,X_{k+l})) + \Sigma_{i\leq j}(-1)^{i+j}\,\omega([X_i,X_j],X_1,..., \hat{X}_i,..., \hat{X}_j,...,X_{k+l})$, where $X_1,...,X_{k+l} \in X(M)$ and $\hat{X}_i$ stands for dropping X_i.*

It can be easily checked that $d : \Omega^k(M) \to \Omega^{k+l}(M)$ satisfies the below conditions:

- d is $\mathbb{R}$–linear,
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)k\omega \wedge d\eta$, for $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$,
- $d \circ d = 0$.

Def. 3.10. *$n \in \mathbb{N}$ is called a local differential dimension of (M, C) at point p, if $\dim T_pM = n$.*

For example one may consider $M = \{n^{-1} \mid n \in \mathbb{N}\} \cup \{0\}$. Then $\dim T_0 = 1$ and $\dim T_pM = 0$ for other points. It is a nice illustrative example of difference with a topological dimension.

Def. 3.11. *$n \in \mathbb{N}$ is called a differential dimension of (M, C) at point p, if for all $p \in M$ $\dim T_pM = n$ and for every $p \in M$ and every $v \in T_pM$ there is a smooth tangent vector field X on (M, C), such that $X(p) = v$. Then (M, C) is called a differential space of constant dimension.*

If (M, C) has constant differential dimension, then set of all global forms is isomorphic to set of all pointwise differential forms. The isomorphism $h_M : A^k(M) \to \Omega^k(M)$ is given by the formula $(h_M\omega)(X_1,...,X_k)(p) := \omega(X_1(p),...,X_k(p))$, where $\omega \in A^k(M), X_1,...,X_k \in X(M), p \in M$, $k = 1,...$ and $h_M := \mathrm{id}_C$ in case of $k = 0$ (Heller et al., 1989)

4. THE CASE OF LAPLACIAN

In a similar way as in Sec. 3 the Laplace operator may be constructed. The Laplace operator, Δf, is a differential operator given in terms of the divergence and the gradient of a function f on the Euclidean space. One way of generalising this operator is to express it in terms of differential forms and Hodge dual (Flanders, 1989). It is then defined by the formula $\Delta f := d * df$, where $*$ denotes Hodge dual.

The Hodge dual operator $* : \Omega^p(M) \to \Omega^{n-p}(M)$ is a linear mapping, therefore completely determined by the formula $\omega \wedge \langle *\eta \rangle = \langle \omega, \eta \rangle\, e1 \wedge \cdots \wedge e_n$, where $\langle \cdot,\cdot \rangle$ denotes inner product. However the inner product may easily be constructed as non–degenerated, symmetric, 2–form on (M, C). The properties of it in the differential spaces category have been studied thoroughly, for example in Heller (1989).

On the other hand the Hodge dual operator may be computed by remembering that it is a linear operator and that:

- $*(1) = e_1 \wedge \cdots \wedge e_n$
- $*(e1 \wedge \cdots \wedge e_n)=1$
- $*(e1 \wedge \cdots \wedge e_k)= \langle e_1,e_1\rangle \ldots \langle e_k,e_k\rangle e_{k+1} \wedge \ldots \wedge e_n$

5. APPLICATION

An easy example of a differential space, which is not a manifold is a cross on $\mathbb{R}^n$, i.e. (M, C), where $M := \{(x, y) \in \mathbb{R}^n \mid xy = 0\}$ and $C := C^\infty|M$. Due to Def. 3.10 the local differential dimension is 1 if $x \neq 0$ or $y \neq 0$, and 2 in othercase. Indeedinpoints $\{(x, y) \in M \mid x \neq 0, y = 0\}$ thetangentspaceis generated by $\frac{\partial}{\partial x}|_{(x,0)}$, and respectively in points $\{(x, y) \in M \mid y \neq 0 , x = 0\}$ is generated by $\frac{\partial}{\partial y}|_{(0,y)}$. Meanwhile $\frac{\partial}{\partial x}|_{(0,0)}, \frac{\partial}{\partial y}|_{(0,0)}$ generate tangent space in point (0, 0).

Equivalently one may think of the above operators as acting on functions evaluated in point sequences. For example in the first case, for an arbitraty $f \in C$, it would be $\frac{\partial f}{\partial x}|_{(x,0)} = \lim_{x_n \to x} \frac{f(x_n,0)-f(x,0)}{|x_n-x|}$. It is also known that in case of $M \subset \mathbb{R}^n$ any tangent vector may be expressed by the above limit formula (Kowalczyk, 1980).

Consider now the von Koch curve and the process of obtaining it from real line by recursively replacing line segments by triangles. Due to the above arguments in each step the number of points in which a tangent space has a local differential dimension 2 is increased. In classical differential manifold language a number of points of non–differentiability increases. But in differential spaces language it is stressed that only local dimension changes. As a fractal, obtained after infinite steps, von Koch curve has everywhere local differential dimension 2 (which is in this case analogous of being nowhere differentiable). So the general constructions from Sec. 4 may be build on von Koch curve considered as a differential space.

Similarly local differential dimension for Sierpinski triangle is 3 except three vertices of the „big", „covering" triangle where it is 2.

6. CONCLUSIONS

It is interesting that by dropping one axiom the whole differential geometry may be studied on objects from some wider category than differential manifolds. Moreover the fact that at least some fractals are differential spaces opens a possibility of further studies. As far as now it seems that such an approach has not been taken.

Finally it is worth to mention that for closed subsets of R^n geometrical objects may be constructed by pullbacks. So by widening the considered category to differential spaces one gains not only a powerful but also a simple tool. Further studies are planned, e.g. on Weierstrass function.

REFERENCES

Barlow, M.T. (1998) Diffusions on fractals, in: M.T. Barlow, D.B.P. Nualart (Eds.), Lectures in Probability Theory and Statistics, Springer, Berlin, pp. 1–121. http://dx.doi.org/10.1007/BFb0092537

Buchner, K., Heller, M., Multarzyński, P., Sasin, W. (1993) Literature on Differential Spaces, Acta Cosmologica 19, 111–129.

Flanders, H. (1989) Differential Forms with Applications to the Physical Sciences, second ed., Dover Publications, New York.

Harrison, J. (1998) Continuity of the integral as a function of the domain, J. Geom. Anal. 8, 769–795. http://dx.doi.org/10.1007/BF02922670

Hambly, B. M. and Kumagai, T. (1999) Transition density estimates for diffusion processes on post critically finite self–similar fractals, Proc. London Math. Soc. 78, 431–458. http://dx.doi.org/10.1112/S0024611599001744

Heller, M., Multarzyński, P., W. Sasin (1989) The algebraic approach to space–time geometry, Acta Cosmologica 16, 53–85.

Jonsson, A. Wallin, H. (1984) Function Spaces on Subsets of Rn, first ed., Harwood Academic Publishers, Chur.

Kigami, J. (1989) A harmonic calculus on the Sierpinski spaces, Japan J. Appl. Math. 6, 259–290. http://dx.doi.org/10.1007/BF03167882

Kigami, J. (2001) Analysis on Fractals, first ed., Cambridge University Press, Cambridge.

Kowalczyk, A. (1980). Tangent Differential Spaces and Smooth Forms, Demonstratio Math. 4, 893–905.

Mosco, U. (1998) Dirichlet forms and self–similarity, in: S.T. Yau (Ed.), New Directions in Dirichlet Forms, AMS, Providence, pp. 117–155.

Strichartz, R.S. (2006) Differential Equations on Fractals, first ed., Princeton University Press, Princeton.

Sikorski, R. (1967) Abstract Covariant Derivative, Colloq. Math. 18, 251–272.