DOI: 10.15415/mjis.2014.31002

Some Faintly Continuous Functions on Generalized Topology

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Received: June 12, 2014| Revised: September 11, 2014| Accepted: September 11, 2014

Published online: September 20, 2014

Abstract: A function $f : (X, \mu) \rightarrow (Y, \sigma)$ is said to be faintly (μ, σ)-continuous if $f^{-1}(V)$ is μ-open in X for every θ-open set V of Y. In this paper the authors introduce and investigate some types of faintly $(\mu\text{-}\sigma)$-continuous functions on generalized topological space (X, μ) into the topological space (Y, σ). Some characterizations and properties of such a type of functions are discussed.

Keywords: Faintly continuous functions; Generalized topology; quasi θ-continuous; faintly $(\mu\text{-}\sigma)$-continuous.

1. Introduction

In topology weak forms of open sets play an important role in the generalization of various forms of continuity, using various forms of open sets, many authors have introduced and studied various types of continuity. In 1961, Levine [10] introduced the notion of weak continuity in topological spaces and obtained a decomposition of continuity. Generalized topology was first introduced by Csaszar [2]. We recall some notions defined in [2,6].

Let X be a set. A subset μ of exp X, Note that exp X denotes the power set of X, is called a generalized topology (GT) on X and (X, μ) is called a generalized topological space [2] (GTS) if μ has the following properties:

(i) $\Phi \in \mu$,
(ii) Any union of elements of μ belongs to μ.

For a GTS (X, μ), the elements of μ are called μ-open sets and the complement of μ-open sets are called-closed sets. Consider X={a,b,c} and μ ={Φ,{a},{b},{a,b}}.The μ-closed sets are X,{b,c},{a,c}and{c}. If A={a,b} then A is not μ-closed.

Mathematical Journal of Interdisciplinary Sciences
Vol. 3, No. 1,
September 2014
pp. 15–22

For A⊂X, we denote by c_μ(A) the intersection of all μ-closed sets containing A, that is the smallest μ-closed set containing A, and by i_μ(A), the union of all μ-open sets contained in A, that is the largest μ-open set contained in A. Intensive research on the field of generalized topological space (X, μ) was done in the past ten years as the theory was developed by A. Csaszar[2], A.P. Dhana Balan[6]. It is easy to observe that and are idempotent and monotonic, where γ:expX→exp X is said to idempotent if and only if A⊆B⊆X implies $\gamma(\gamma(A))=\gamma(A)$ and monotonic if and only if A⊆B⊆X implies $\gamma(A) \subseteq \gamma(B)$. It is also well known that from [4,5] that if μ is a GT on X and A ⊆ X, x∈X then $x \in c_\mu(A)$ if and only if $x \in M \in \mu \Rightarrow M \cap A \neq \Phi$ and $c_\mu(X-A) = X-i_\mu(A)$.

Let B ⊂expX and Φ ∈ B. Then B is called a base[3] for μ if $\{\cup B' : B' \subset B\} = \mu$.We also say that μ is generated by B. A point x∈X is called a μ-cluster point of B⊂X if U∩ (B-{x}) ≠ Φ for each U ∈ μ with $x \in \mu$. The set of all μ-cluster point of B is denoted by d(B). A GT μ is said to be strong [3] if X ∈ μ. Throughout this paper, a space (X, μ) or simply X for short, will always mean a strong generalized topological space with strong generalized topology μ and (Y, σ) to be a topological space, unless otherwise explicitly stated. The end or omission of a proof will be denoted by ■.

Let (Y, σ) be a topological space with topology σ. Let A⊂(*Y*, σ). The closure and interior of A is denoted by *cl*(A) and *int*(A) respectively, where the closure of A is the intersection of all closed sets containing A and the interior of A is the union of all open sets contained in A.

Let (X,τ) be a topological Space. The δ-closure[22] of a subset A of a topological space (X, τ) is defined by$\{x \in X : A \cap U \neq \Phi$ for all regular open set U containing $x\}$,where a subset A is called regular open if $A = int(cl(A))$. A subset A of a topological space (X, τ) is called semi-open[11] (resp., pre-open[13], α-open[15], β-open[14], b-open[1], δ-pre open [19], δ-semi open[18], and e-open[7]) if $A \subseteq cl(int(A))$ (resp., $A \subseteq int(cl(A))$, $A \subseteq int(cl(int(A))$, $A \subseteq cl(int(cl(A)))$, $A \subseteq cl(int(A)) \cup int(cl(A))$, $A \subseteq int(cl_\delta(A))$, $A \subseteq cl(int_\delta(A))$ and $A \subseteq int(cl_\delta(A)) \cup cl(int_\delta(A))$). A point $x \in X$ is in $s.cl(A)$ (resp., $pcl(A)$) if for each semi open (resp., pre open) set U containing x, $U \cap A \neq \Phi$. A point $x \in X$ is called a θ-cluster[22] (resp., semi θ-cluster[12],P(θ)-cluster[17]) point of A denoted by $cl_\theta(A)$ (resp., $s.\ cl_\theta(A)$, p(θ)-$cl(A)$) if $cl(A) \cap U \neq \Phi$ (resp., $s.cl(A) \cap U \neq \Phi$, p.$cl(A) \cap U \neq \Phi$) for every open (resp., semi-open, pre-open) set U containing x. A subset A is called θ-closed (resp., semi θ-closed, P(θ)-closed) if $cl_\theta(A) = A$ (resp., $s.\ cl_\theta(A)) = A$, p(θ)-$cl(A) = A$). The complement of a θ-closed (resp., semi-θ-closed, p(θ)-closed) set is called θ-open(resp., semi-θ-open, p(θ)-open).The family of all θ-open sets in a topological space forms a topology which is weaker than the original topology. For any topological space(X, τ), the collection of all semi open (resp., pre-open, α-open, β-open, b-open, e-open, θ-open, p(θ)-open) sets are denoted by so(X) (resp., po(X), α-o(X), β-o(X), Bo(X), eo(X), θ-o(X), p θ-o(X)). We note that each of these collections forms a generalized topology on (X, τ).

Recall that, a subset A of (X, τ) is θ-open if for each $x \in A$ there exists an open set U such that $x \in U \subset cl(U) \subset A$.

Definition 1.1 A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(i) faintly continuous if f^{-1}(V) is open in X for every θ-open set V of Y.

(ii) quasi θ-continuous[9] if f^{-1}(V) is θ-open in X for every θ-open set V of Y.

(iii) strongly θ-continuous[16] f^{-1}(V) is θ-open in X for every open set V of Y.

(iv) θ-continuous [8] if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(cl(U)) \subset cl(V)$.

(v) continuous if f^{-1}(V) is open in X for every open set V of Y.

2. Faintly (μ-σ) continuous functions

Definition 2.1 A function $f : (X, \mu) \to (Y, \sigma)$ is said to be faintly (μ-σ)-continuous if for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists μ-open set U of X containing x such that $f(U) \subset V$.

Definition 2.2 A function $f : (X, \mu) \to (Y, \sigma)$ is said to be (μ-σ)-continuous[20] (resp., weakly (μ-σ)-continuous, almost (μ-σ)-continuous) if for each $x \in X$ and each open set V of Y containing $f(x)$ there exists μ-open set U of X containing x such that $f(U) \subset V$ (resp., $f(U) \subset cl(V)$, $f(U) \subset int(cl(v))$).

A faintly (μ-σ)-continuous function $f : (X, \mu) \to (Y, \sigma)$ reduces to a faintly continuous if (μ-τ).

Results 2.3

(i) Every (μ-σ)-continuous function is faintly (μ-σ)-continuous.

(ii) Every (μ-σ)-continuous function is almost (μ-σ)-continuous.

(iii) Every almost (μ-σ)-continuous function is weakly (μ-σ)-continuo us.

The converse need not be true.

Example 2.4 (i) Let X = {a,b,c} and

Let μ ={Φ, x, {a,b}, {b, c}}

Let Y = {a,b,c} and Let σ = {Φ, x, {a}, {a,b}}

Define $f : (X, \mu) \to (Y, \sigma)$ by f(a) = a = f(c), f(b) = b.

Then f is faintly (μ-σ)-continuous but not (μ-σ)-continuous.

ii) Let X = {a,b,c}, μ {Φ, x, {a}, {a,b}, {b,c}} and

Let $Y = \{a,b,c\}$, $\sigma = \{\Phi , x, \{a\}, \{c\}, \{a,c\}, \{b,c\}\}$
Define $f : (X, \mu) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$.
Then f is almost (μ-σ)-continuous but not (μ-σ)-continuous.

iii) Let $X = Y = \{a,b,c\}$ and
Let $\mu = \{\Phi, x, \{a\} \{a,b\}, \{b,c\}$ and
Let $\sigma = \{\Phi, x, \{a\}, \{c\}, \{a,c\}\}$.
Define the identity map $f : (X, \mu) \to (Y, \sigma)$.
Then f is not almost (μ-σ)-continuous but weakly (μ-σ)-continuous.

Theorem 2.5 Let $f : (X, \mu) \to (Y, \sigma)$ be a function. The following statements are equivalent.

(i) f is faintly (μ-σ)-continuous.
(ii) for each θ-open set V of Y, $f^{-1}(V)$ is μ-open in X.
(iii) for each θ-closed set K of Y, $f^{-1}(K)$ is μ-closed in X.
(iv) for each $A \subset X$, $f(c_\mu(A)) \subset cl_\delta f(A)$.

Proof:
(i) ⇒ (ii) If V is θ-open in Y, then for each $x \in f^{-1}(V)$, V is a θ-open set of $f(x)$. Hence, by faintly (μ-σ)-continuity of f, there is a μ-open set U containing x such that $f(U) \subset V$; that is $U \subset f^{-1}(V)$. Thus $f^{-1}(V)$ contains a μ-open set of each of its points and so $f^{-1}(V)$ is μ-open in X.

(ii) ⇒ (iii) Let K be a θ-closed set of Y. Then $f^{-1}(Y\text{-}K)$ is μ-open in X, by part (ii). Hence, since $f^{-1}(K) = X\text{-} f^{-1}(Y\text{-}K)$, $f^{-1}(K)$ is μ-closed in X.

(iii) ⇒ (iv) Let K be any θ-closed set in Y containing $f(A)$. By part (iii), $f^{-1}(K)$ is a μ-closed set in X containing A. Hence, $c_\mu(A) \subset f^{-1}(K)$ and it follows that $f(c_\mu(A)) \subset K$. Since this is true for any θ-closed set K containing $f(A)$, we have $f(c_\mu(A)) \subset cl_\delta f(A)$.

(iv) ⇒ (i) Let $x \in X$ and Let V be a θ-open set of $f(x)$. Set $A = X - f^1(V)$ and Let $U = X - (A)$. Since $f(c_\mu(A)) \subset cl_\delta f(A)$. We have, $f(c_\mu(A)) = f(c_\mu(X - f^{-1}(V))) \subset cl_\sigma f(A)$ That is, $f(X\text{-}U) \subset cl_\delta f(A)$ and so $x \in U$. That is even clear that $f(U) \subset V$. Hence f is faintly (μ-σ)-continuous. ■

Theorem 2.6 Let $f:(X,\mu)\to(Y,\sigma)$ be faintly $(\mu-\sigma)$-continuous and Let $g:(Y,\sigma)\to(Z,\gamma)$ be quasi--continuous. Then $g^\circ f: X \to Z$ is faintly $(\mu-\gamma)$-continuous.

Proof : Let V be θ-open in Z. Then $g^{-1}(V)$ is θ-open in Y by quasi-θ-continuity of g. Hence, by faintly (μ-σ)-continuity of f, $f^{-1}(g^{-1}(V))$ is μ-open in X. But

$f^{-1}(g^{-1}(V)) = (f \circ g)^{-1}$ (V)which is μ-open in X. Thus $g \circ f$ is faintly $(\mu - \gamma)$ -continuous. ■

Theorem 2.7 The following statements hold for functions $f : (X, \mu) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \gamma)$:

(i) If f is (μ-σ)-continuous and g is quasi-θ-continuous, then g°f : $X \to Z$ is faintly (μ–γ)-continuous.
(ii) If f is faintly (μ-σ)-continuous and g is strongly θ-continuous, then g°f is (μ–γ)-continuous.
(iii) If f is (μ-σ)-continuous and g is strongly -continuous, then g°f is (μ–γ)-continuous.
(vi) If f is (μ-σ)-continuous and g is faintly continuous, then g°f is faintly (μ–γ)-continuous.
(v) If f is weakly (μ-σ)-continuous and g is continuous, then g°f is weakly (μ–γ)-continuous.
(vi) If f is almost (μ-σ)-continuous and g is continuous, then g°f is almost (μ–γ)-continuous.

Definition 2.8 A GTS (X, μ) is said to be

(i) μ-regular [21] if for each μ-closed set F and each point $x \notin$F, there exists disjoint μ-open sets U and V such that $x \in$U, F⊆ V.
(ii) μ-normal [21] if for any two disjoint μ-closed subsets F and K, there exists disjoint μ-open sets U and V such that F⊆U, K⊆V.

Lemma 2.9 If a mapping $f : X \to Y$ is (μ-σ)-closed, then for each subset B of Y and each μ-open set U in X containing $f^{-1}(B)$, there exists an open set V in Y containing B such that f^{-1}(V) $\subset U$.

Theorem 2.10 Let $f : (X, \mu) \to (Y, \sigma)$ be a (μ-σ)-continuous, (μ-σ)-open and surjective. If X is μ-regular, then Y is regular.

Proof: Let F be closed in Y and y $\notin$ F. Then f^{-1}(F) ∩ f^{-1}(y) ≠ Φ and f^{-1}(F) is μ-closed in X. Take a point $x \in f^{-1}$(y). Since X is μ-regular, there exists disjoint open sets U_1 and U_2 such that $x \in U_1$ and f^{-1}(F) ⊂ U_2. By lemma 2.9, since f is (μ-σ)-continuous, (μ-σ)-open, there exists an open set V in Y such that F ⊂ V and (V) ⊂ U_2. Since f is (μ-σ)-continuous and (μ-σ)-open, we have $f(U_1)$ is open in Y. Since U_1 and U_2 are disjoint, we have $f(U_1)$ ∩ V ≠ Φ. Hence Y is regular. ■

Theorem 2.11 Let $f : (X, \mu) \to (Y, \sigma)$ be a (μ-σ)-continuous, (μ-σ)-open and surjective. If X is μ-normal, then Y is normal.

Proof: If F_1 and F_2 are disjoint closed sets in *Y*, then by the (μ-σ)-continuity of *f*, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint μ-closed sets of X. Since *X* is μ-normal, there exists disjoint μ-open sets U_1 and U_2 in *X* such that $f^{-1}(F_i) \subset U_i, i = 1,2$. Then by lemma 2.9, there exists open sets V_1 and V_2 in Y such that $F_i \subset V_i$ and $f^{-1}(V_i) \subset U_i$, i=1,2. Since *f* is (μ-σ)-open and surjective and $U_1 \cap U_2 \neq \Phi$, we obtain $V_1 \cap V_2 \neq \Phi$. This shows that *Y* is normal. ■

Definition 2.12 Let $f:(X,\mu) \to (Y,\sigma)$ and Let A ⊂ X. The restriction of *f* to A by *f*/A is to denote the map of A into *Y* defined by (*f*/A)(a) = *f*(a) for each a ∈ A.

Theorem 2.13 Let (X, μ) be a GTS. If A ⊂ *X* and *f*:(*X*, μ) → (*Y*, σ) is (μ-σ)-continuous, then (*f*/A): A → *Y* is (μ-σ)-continuous.

Proof: Let V be open in Y. Then $(f/A)^{-1}(V) = (f/A)^{-1}(V) \cap A$ and $f^{-1}(V) \cap A$ is μ-open in the relative topology on A. ■

Theorem 2.14 Let (*X*, μ) be a GTS. If A ⊂ *X* and *f*:(*X*, μ) → (*Y*, σ) is faintly (μ-σ)-continuous, then (*f*/A): A → Y is a faintly (μ-σ)-continuous.

Proof: Let V be θ-open in *Y*. Then $(f/A)^{-1}(V) = f^{-1}(V) \cap A$ and $f^{-1}(V) \cap A$ is μ-open in the relative topology on A. ■

Theorem 2.15 If $X = A \cup B$, where A and B are both μ-open (or both μ-closed) in *X* and if *f*: (*X*, μ) → (*Y*, σ) is a function such that both *f*/A and *f*/B are faintly (μ-σ)-continuous then *f* is faintly (μ-σ)-continuous.

Proof: Suppose A and B are -open. Suppose V is μ-open in *Y*. Then (V) is θ-open in *X*, since $f^{-1}(V) = (f/A)^{-1}(V) \cup (f/B)^{-1}(V)$. Also $(f/A)^{-1}(V)$ and $(f/B)^{-1}(V)$ are μ-open in an μ-open subspace of *X* and so μ-open in *X*.

If A and B are μ-closed, Proof is similar. ■

Theorem 2.16 Let Y be a subspace of the topological space Z containing the image set *f*(*x*) and *f*:(*X*, μ) → (*Y*, σ) is faintly (μ-σ)-continuous, then *f*:(*X*, μ) → (*Z*, γ) is faintly (μ-σ)-continuous as a map from *X* to Z.

Proof: Let *f*:(*X*, μ) → (*Y*, σ) be faintly (μ-σ)-continuous. If *f*(x) ⊂ Y ⊂ Z, we show that the function *f*:(*X*, μ) → (Z, γ) is faintly (μ-σ)-continuous. Let B be θ-open in *Y*. Then B = $Y \cap U$ for some θ-open set *U* of Z as the set of all θ-open set forms a topology and *Y* is a subspace of Z. Since *Y* contains the entire image set of *f*(*x*), we have $f^{-1}(B) = f^{-1}(Y \cap U) = f^{-1}(Y) \cap f^{-1}(U) = f^{-1}(U)$ which is μ-open in *X*. Thus *f*: *X* → Z is faintly (μ-σ)-continuous.

Let *f* : *X* → Y be a function. A function g : *X* → *X* x *Y* defined by g(*x*) = (*x*, *f*(*x*)) for every $x \in X$, is called the graph function of *f*. ■

Definition 2.17 A function f:(X, μ) → (Y, σ) is faintly μ-closed graph if for each (x,y) ∈ (X x Y) = G(f) there exist $U \in \mu$ containing x and a θ-open set V in Y containing y such that (U x V) ∩ G(f) ≠ Φ.

Theorem 2.18 A function f:(X, μ) → (Y, σ) is faintly (μ-σ)-continuous if the graph function is faintly (μ-σ)-continuous.

Proof: Let $x \in X$ and Let V be a θ-open set containing $f(x)$. Then X x V is θ-open in X x Y and contains g(x) = (x, $f(x)$). Therefore, there exists $U \in \mu$ containing x such that g(U) ⊂ X x V. This implies $f(U)$ ⊂ V. Thus f is faintly (μ-σ)-continuous.

The following result is obvious from the definition 2.17. ■

Result 2.19 The graph G(f) of a function f:(X, μ) → (Y, σ) is faintly μ-closed if and only if for each (x,y) ∈ (X x Y) – G(f) there exist U ∈ μ containing x and a θ-open set V in Y containing y such that $f(U)$ ∩ V ≠ Φ.

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