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Some Inequalities for Polynomials Vanishing Inside a Circle

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Abstract: If $P(z)=\Sigma_{j=0}^{n}a_jz^j$ is a polynomial of degree n having all its zeros in $|z|\leq k, k\geq 1$, then it is known that

$$Max_{|z|=1}|P'(z)|\geq \frac{n}{1+k^n}Max_{|z|=1}|P(z)|$$

In this paper we obtain a generalization as well as an improvement of the above inequality. Besides this some other results are also obtained.

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1. Introduction and Statement of Results

If $P(z)$ is a polynomial of degree n, then concerning the estimate of $|P'(z)|$, it was shown by Turan [8] that if $P(z)$ has all its zeros in $|z|\leq 1$, then

$$Max_{|z|=1}|P'(z)|\geq \frac{n}{2}Max_{|z|=1}|P(z)| \tag{1}$$

In (1) equality holds if all the zeros of $P(z)$ lie in $|z| = 1$.

More generally if the polynomial P(z) has all its zeros in $|z| \leq k \leq 1$, it was proved by Malik [7] that the inequality (1) can be replaced by

$$Max_{|z|=1}|P'(z)|\geq \frac{n}{1+k}Max_{|z|=1}|P(z)| \tag{2}$$

The case when P(z) has all its zeros in $|z|\leq k, k\geq 1$, was setteled by Govil [5], who proved

Theorem A. If $P(z)=\Sigma_{j=0}^{n}a_j z^j$ is a polynomial of degree n havihg all its zeros in $|z| \leq k, k \geq 1$, then

$$Max_{|z|=1}|P'(z)| \geq \frac{n}{1+k^n} Max_{|z|=1}|P(z)| \tag{3}$$

The result is best possible and equality holds for the polynomial $P(z) = z^n + k^n$.

Aziz [1] improved upon the bound in (3) by taking into account the location of all the zeros of the polynomial $P(z)$ instead of concerning only the zero of largest modulus. More precisely, he proved the following result.

Theorem B. If all the zeros of the polynomial $P(z)=\Pi_{j=1}^{n}(z-z_j)$ of degree n lie in $|z| \leq k$, $k \geq 1$, then

$$Max_{|z|=1}|P'(z)| \geq \frac{2}{1+k^n}\left(\sum_{j=1}^{n}\frac{k}{k+|z_j|}\right) Max_{|z|=1}|P(z)| \tag{4}$$

The result is best possible and equality holds in (4) for $P(z) = z^n + k^n$.

Govil [3] also obtained the following improvement of Theorem B.

Theorem C. Let $P(z)=\sum_{j=0}^{n}a_j z^j = a_n \prod_{j=1}^{n}(z-z_j)$, $a_n \neq 0$, be a polynomial of degree $n \geq 2, |z_j| \leq k_j, 1 \leq j \leq n$, and let $k = max\{k_1, k_2, \ldots, k_n\} \geq 1$. Then

$$\begin{aligned} Max_{|z|=1}|p'(z)| \geq & \frac{2}{1+k^n}\left(\sum_{j=1}^{n}\frac{k}{k+k_j}\right) \max_{|z|=1}|P(z)| \\ & + \frac{2|a_{n-1}|}{1+k^n}\sum_{j=1}^{n}\frac{1}{k+k_j}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right) \\ & + |a_1|\left(1-\frac{1}{k^2}\right) \end{aligned} \tag{5}$$

if $n > 2$ and

$$\begin{aligned} Max_{|z|=1}|p'(z)| \geq & \frac{2}{1+k^n}\left(\sum_{j=1}^{n}\frac{k}{k+k_j}\right) \max_{|z|=1}|P(z)| \\ & + \frac{(k-1)^n}{1+k^n}|a_1|\sum_{j=1}^{n}\frac{1}{k+k_j} \\ & + \left(1-\frac{1}{k}\right)|a_1| \, if \, n=2. \end{aligned} \tag{6}$$

In (5) and (6) equality holds for $P(z) = z^n + k^n$.

In this paper we prove the following result which gives an improvement of Theorem C, and thus as well of inequality (4).

Theorem D. Let $P(z)=\sum_{j=0}^{n} a_j z^j = a_n \prod_{j=1}^{n}(z-z_j),\ a_n \neq 0$ be a polynomial of degree $n \geq 2$ and $P(0)\neq 0, |z_j| \leq k_j,\ 1 \leq j \leq n$, and let $k = max\{k_1, k_2, \ldots, k_n\} \geq 1$. Then

$$\begin{aligned} Max_{|z|=1}|P'(z)| \geq {} & \frac{2}{1+k^n}\left(\sum_{j=1}^{n}\frac{k}{k+k_j}\right)Max_{|z|=1}|P(z)| \\ & + \frac{k^n-1}{k^n(1+k^n)}\left(\sum_{j=1}^{n}\frac{k}{k+k_j}\right)\min_{|z|=k}|P(z)| \\ & + \frac{k|P'(0)|+k^{n-1}|Q'(0)|}{k^n(1+k^n)}\left(\sum_{j=1}^{n}\frac{k}{k+k_j}\right)\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right) \\ & + \left(1-\frac{1}{k^2}\right)|P'(0)| \ if\ n>2. \end{aligned} \tag{7}$$

and

$$\begin{aligned} Max_{|z|=1}|P'(z)| \geq {} & \frac{2}{1+k^n}\left(\sum_{j=1}^{n}\frac{k}{k+k_j}\right)Max_{|z|=1}|P(z)| \\ & + \frac{k^n-1}{k^n(1+k^n)}\left(\sum_{j=1}^{n}\frac{k}{k+k_j}\right)\min_{|z|=k}|P(z)| \\ & + \frac{k|P'(0)|+k^{n-1}|Q'(0)|}{k^n(1+k^n)}\left(\sum_{j=1}^{n}\frac{k}{k+k_j}\right)\frac{(k-1)^n}{n} \\ & + \left(1-\frac{1}{k}\right)|P'(0)| \ if\ n=2. \end{aligned} \tag{8}$$

where $Q(z)=z^n\overline{P\left(\frac{1}{\overline{z}}\right)}$.

The result is best possible and equality holds in (7) and (8) for $P(z) = z^n + k^n$.

Since $\frac{k}{k+k_j} \geq \frac{1}{2}$ for $1 \leq j \leq n$, then above theorem gives in particular.

Corollary 1. If $P(z)=a_n\Pi_{j=1}^{n}(z-z_j), a_n \neq 0$, is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k$, $k \geq 1$, $P(0) \neq 0$, then

$$Max_{|z|=1}|P'(z)| \geq \frac{n}{1+k^n} Max_{|z|=1}|P(z)| + \frac{n(k^n-1)}{2k^n(1+k^n)} \min_{|z|=k}|P(z)|$$
$$+\frac{n(k|P'(0)|+k^{n-1}|Q'(0)|)}{2k^n(1+k^n)}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)$$
$$+\left(1-\frac{1}{k^2}\right)|P'(0)| \; if\ n>2, \tag{9}$$

and

$$Max_{|z|=1}|P'(z)| \geq \frac{n}{1+k^n} Max_{|z|=1}|P(z)| + \frac{n(k^n-1)}{2k^n(1+k^n)} \min_{|z|=k}|P(z)|$$
$$+\frac{(k-1)^n(k|P'(0)|+k^{n-1}|Q'(0)|)}{2k^n(1+k^n)} \tag{10}$$
$$+\left(1-\frac{1}{k}\right)|P'(0)| \; if\ n=2,$$

where $Q(z)=z^n\overline{P\left(\frac{1}{\overline{z}}\right)}$.

Equality holds in (9) and (10) for $P(z) = z^n + k^n$.

It is easy to verify that if $k > 1$ and $n > 2$, then $\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)>0$, hence for polynomials of degree > 1, (9) and (10) together provide a refinement of Theorem A.

Now, on applying above theorem to $Q(z)=z^n\overline{P\left(\frac{1}{\overline{z}}\right)}$, we obtain the following result which gives a generalization as well as an improvement upon some well known inequalities.

Corollary 2. If $P(z)=a_n\Pi_{j=1}^n(z-z_j), a_n\neq 0$, is a polynomial of degree $n \geq 2$ and $|z_j| \geq k_j$ and let $k = min\{k_1, k_2, ..., k_n\} \leq 1$, then if $|P'(z)|$ and $|Q'(z)|$ attain the maximum at the same point on $|z| = 1$, we have

$$Max_{|z|=1}|P'(z)| \leq \frac{1}{1+k^n}\left(n-k^n\sum_{j=1}^{n}\frac{k_j-k}{k_j+k}\right) Max_{|z|=1}|P(z)|$$
$$-\left(\frac{1-k^n}{1+k^n}\right)\left(\sum_{j=1}^{n}\frac{k_j}{k+kj}\right)\min_{|z|=k}|P(z)|$$

$$-\frac{(k\,|\,P'(0)\,|+k^{n-1}\,|\,Q'(0)\,|)}{1+k^n}\left(\sum_{j=1}^{n}\frac{k_j}{k+kj}\right)\left(\frac{1-k^n}{n}-\frac{k^2(1-k^{n-2})}{n-2}\right) \quad (11)$$
$$-(1-k^2)\,|\,Q'(0)\,|\,if\ n>2,$$

and

$$Max_{|z|=1}\,|\,P'(z)\,|\le\frac{1}{1+k^n}\left(n-k^n\sum_{j=1}^{n}\frac{k_j-k}{k_j+k}\right)Max_{|z|=1}\,|\,P(z)\,|$$
$$-\left(\frac{1-k^n}{1+k^n}\right)\left(\sum_{j=1}^{n}\frac{k_j}{k+kj}\right)\min_{|z|=k}\,|\,P(z)\,| \quad (12)$$
$$-\frac{(k\,|\,P'(0)\,|+k^{n-1}\,|\,Q'(0)\,|)}{1+k^n}\left(\sum_{j=1}^{n}\frac{k_j}{k+kj}\right)\frac{(1-k)^n}{n}$$
$$-(1-k)\,|\,Q'(0)\,|\,if\ n=2,$$

where $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$.

The result is best possible for $k_j=k, 1\le j\le n$ and equality holds in (11) and (12) for $P(z) = z^n + k^n$.

Proof of Corollary 2. Since the zeros of *P(z)* satisfy $|\,z_j\,|\ge k_j, 1\le j\le n$, such that $k=min\{k_1,k_2,\ldots,k_n\}$. It follows that the zeros of $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$ satisfy $\frac{1}{z_j}\le\frac{1}{k_j}, 1\le j\le n$, such that $\frac{1}{k}=\max\left(\frac{1}{k_1},\frac{1}{k_2},\ldots,\frac{1}{k_n}\right)$ or equivalently k = $min\{k_1, k_2,...,k_n\}$. Applying inequality (7) to the polynomial $Q(z)$ and for the case $n > 2$, we have

$$Max_{|z|=1}\,|\,Q'(z)\,|\ge\frac{2}{1+\left(\frac{1}{k}\right)^n}\left(\sum_{j=1}^{n}\frac{\frac{1}{k}}{\frac{1}{k}+\frac{1}{k_j}}\right)Max_{|z|=1}\,|\,Q(z)\,|$$
$$+\frac{\left(\frac{1}{k^n}-1\right)}{\left(1+\frac{1}{k^n}\right)\frac{1}{k^n}}\left(\sum_{j=1}^{n}\frac{\frac{1}{k}}{\frac{1}{k}+\frac{1}{k_j}}\right)\min_{|z|=\frac{1}{k}}\,|\,Q(z)\,|$$

$$+\frac{\left(\frac{1}{k}|Q'(0)|+\frac{1}{k^{n-1}|P'(0)|}\right)}{\left(1+\frac{1}{k^n}\right)\frac{1}{k^n}}\left(\sum_{j=1}^{n}\frac{\frac{1}{k}}{\frac{1}{k}+\frac{1}{k_j}}\right)$$

$$\times\left(\frac{\left(\frac{1}{k}\right)^n-1}{n}-\frac{\left(\frac{1}{k}\right)^{n-2}-1}{n-2}\right)+\left(1-\frac{1}{\left(\frac{1}{k}\right)^2}\right)|Q'(0)|$$

$$=\frac{2k^n}{1+k^n}\left(\sum_{j=1}^{n}\frac{k}{k+k_j}\right)Max_{|z|=1}|P(z)|$$

$$+\frac{k^n(k|P'(0)|+k^{n-1}|Q'(0)|}{(1+k^n)}\left(\sum_{j=1}^{n}\frac{k_j}{k+k_j}\right)$$

$$\times\left(\frac{1-k^n}{nk^n}-\frac{1-k^{n-2}}{(n-2)k^{n-2}}\right)+(1-k^2)|Q'(0)|. \tag{13}$$

Let $Max_{|z|=1}|P'(z)=|P'(e^{i\alpha})|$, where $0 \le \alpha \le 2\pi$. Since $|P'(z)|$ and $|Q'(z)|$ attains their maximum at the same point on $|z| = 1$, it follows that $Max_{|z|=1}|Q'(z)=|Q'(e^{i\alpha})|$.

We have by Lemma 1(stated in section 2) that

$$|P'(e^{i\alpha})|+|Q'(e^{i\alpha})|\le nMax_{|z|=1}|P(z)| \tag{14}$$

Using (13) in (14), we get

$$nMax_{|z|=1}|P(z)|\ge|P'(e^{i\alpha})|+\frac{2k^n}{1+k^n}\left(\sum_{j=1}^{n}\frac{k_j}{k+k_j}\right)Max_{|z|=1}|P(z)|$$

$$+\left(\frac{1-k^n}{1+K^n}\right)\sum_{j=1}^{n}\frac{k_j}{k+k_j}\min_{|z|=k}|P(z)|$$

$$+\frac{(k|P'(0)|+k^{n-1}|Q'(0)|)}{(1+k^n)}\sum_{j=1}^{n}\frac{k_j}{k+k_j}\left(\frac{1-k^n}{n}-\frac{k^2(1-k^{n-2})}{n-2}\right)$$

$$+(1-k^2)|Q'(0)|$$

implying

$$
\begin{aligned}
|P'(e^{i\alpha})| \leq & \left(n - \frac{2k^n}{1+k^n}\sum_{j=1}^{n}\frac{k_j}{k+k_j}\right) Max_{|z|=1}|P(z)| \\
& -\left(\frac{1-k^n}{1+K^n}\right)\sum_{j=1}^{n}\frac{k_j}{k+k_j}\min_{|z|=k}|P(z)| \\
& -\frac{(k|P'(0)|+k^{n-1}|Q'(0)|)}{(1+k^n)}\sum_{j=1}^{n}\frac{k_j}{k+k_j}\left(\frac{1-k^n}{n} - \frac{k^2(1-k^{n-2})}{n-2}\right), \\
& -(1-k^2)|Q'(0)|
\end{aligned}
$$

which proves the desired inequality for $n > 2$.

The result for $n = 2$ follows on the same lines as that for $n > 2$ but instead of using (7), we use (8).

As it is easy to see, our Corollary 2 provides a generalization as well as an improvement of the following result due to A. Aziz and N.Ahmad [2].

Theorem D. Let $P(z) = \Pi_{j=1}^{n}(z - z_j)$ be a polynomial of degree n which does not vanish in $|z| < k$, where $k \leq 1$, and let $Q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$, then

$$Max_{|z|=1}|P'(z)| \leq \frac{1}{1+k^n}\left(n - k^n\sum_{j=1}^{n}\frac{|z_j|-k}{|z_j|+k}\right)Max_{|z|=1}|P(z)|. \tag{15}$$

The result is best possible and equality holds for the polynomial $P(z) = z^n + k^n$.

2. Lemmas

For the proof of the theorem, we need the following lemmas.

Lemma 1. If $P(z)$ is a polynomial of degree n, then on $|z| = 1$

$$|P'(z)| + |Q'(z)| \leq nMax_{|z|=1}|P(z)| \tag{16}$$

where $Q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$.

This is a special case of a result due to Govil and Rahman [6, *Lemma*10]. Lemma 2. If $P(z)$ is a polynomial of degree n, then for $R > 1$,

$$Max_{|z|=R}|P(z)| \leq R^n Max_{|z|=1}|P(z)| - (R^n - R^{n-2})|P(0)| \text{ for n} > 1, \tag{17}$$

and

$$Max_{|z|=R}|P(z)| \leq RMax_{|z|=1}|P(z)| - (R-1)|P(0)| \text{ for n} = 1. \tag{18}$$

The above result is due to Frappier, Rahman and Ruscheweyh [3, *Theorem*2].
Lemma 3. If $P(z)$ is a polynomial of degree $n \geq 2$ and $P(0) \neq 0$, then for $|z| = 1$ and $R \geq 1$,

$$\begin{aligned} &|P(Rz)-P(z)|+|Q(Rz)-Q(z)| \\ &\qquad \leq (R^n-1)Max_{|z|=1}|P(z)| \\ &\qquad -(|P'(0)|+|Q'(0)|)\left(\frac{R^n-1}{n}-\frac{R^{n-2}-1}{n-2}\right) for \;\; n>2, \end{aligned} \tag{19}$$

and

$$\begin{aligned} &|P(Rz)-P(z)|+|Q(Rz)-Q(z)| \\ &\qquad \leq (R^n-1)Max_{|z|=1}|P(z)| \\ &\qquad -(|P'(0)|+|q'(0)|)\frac{(R-1)^n}{n} for \;\; n=2, \end{aligned} \tag{20}$$

where $Q(z)=z^n\overline{P\left(\frac{1}{\overline{z}}\right)}$.

Proof of Lemma 3. If $P(z)$ is a polynomial of degree n and $p(0) \neq 0$, then $Q(z)=z^n\overline{P\left(\frac{1}{\overline{z}}\right)}$ is also a polynomial of degree n. Applying inequality (17) of Lemma 2 to the polynomials $P'(z)$ and $Q'(z)$, which are of degree (n — 1), we obtain for all $t \geq 1$ and $0 \leq \theta < 2\pi$,

$$|P'(te^{i\theta})| \leq t^{n-1}|P'(e^{i\theta})|-(t^{n-1}-t^{n-3})|P'(0)| \; for \;\; n>2 \tag{21}$$

and

$$|Q'(te^{i\theta})| \leq t^{n-1}|Q'(e^{i\theta})|-(t^{n-1}-t^{n-3})|Q'(0)| \; for \;\; n>2. \tag{22}$$

Adding inequalities (21) and (22), we get for $n > 2$,

$$\begin{aligned} |P'(te^{i\theta})|+|Q'(te^{i\theta})| &\leq t^{n-1}\left(|P'(e^{i\theta})+|Q'(e^{i\theta})\right) \\ &\quad -(t^{n-1}-t^{n-3})\left(|P'(0)|+|Q'(0)|\right). \end{aligned}$$

Using Lemma 1 in above inequality, we get

$$\begin{aligned} |P'(te^{i\theta})|+|Q'(te^{i\theta})| &\leq nt^{n-1}Max_{|z|=1}|P(z)| \\ &\quad -(t^{n-1}-t^{n-3})\left(|P'(0)|+|Q'(0)|\right) \end{aligned} \tag{23}$$

for $0 \leq \theta < 2\pi$ and $t \geq 1$.

Also for each $\theta, 0 \le \theta < 2\pi$ and $R > 1$, we have

$$P(\mathrm{Re}^{i\theta}) - P(e^{i\theta}) = \int_1^R e^{i\theta} P'(te^{i\theta})dt$$

and

$$Q(\mathrm{Re}^{i\theta}) - Q(e^{i\theta}) = \int_1^R e^{i\theta} Q'(te^{i\theta})dt$$

Hence for $n > 2$

$$|P(\mathrm{Re}^{i\theta}) - P(e^{i\theta})| + |Q(\mathrm{Re}^{i\theta}) - Q(e^{i\theta})| \le \int_1^R (|P'(te^{i\theta})| + |Q'(te^{i\theta})|)dt.$$

On combining inequality (23) with the above inequality, we get for $n > 2$

$$\begin{aligned} &|P(\mathrm{Re}^{i\theta}) - P(e^{i\theta})| + |Q(\mathrm{Re}^{i\theta}) - Q(e^{i\theta})| \\ &\le \int_1^R [nt^{n-1} Max_{|z|=1} |P(z)| - (t^{n-1} - t^{n-3})(|P'(0)| + |Q'(0)|]dt \\ &= (R^n - 1) Max_{|z|=1} |P(z)| - (|P'(0)| + |Q'(0)|)\left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right) \end{aligned}$$

for $0 \le \theta < 2\pi$ and $R > 1$, which is equivalent to the desired result.

The proof of (20) follows on the same lines as the proof of (19), but instead of using (17), it uses (18).

Lemma 4. If $P(z)$ is a polynomial of degree $n \ge 2$ having all its zeros in $|z| \le k, k \ge 1$, and $P(0) \ne 0$, then

$$\begin{aligned} Max_{|z|=k} |P(z)| \ge \frac{2k^n}{1+k^n} & Max_{|z|=1} |P(z)| \\ & + \frac{(k|P'(0)| + k^{n-1}|q'(0)|)}{(1+k^n)} \\ & \times \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n-2}\right) for\ n > 2, \end{aligned} \tag{24}$$

and

$$\begin{aligned} Max_{|z|=k} |P(z)| \ge \frac{2k^n}{1+k^n} & Max_{|z|=1} |P(z)| \\ & + \frac{(k|P'(0)| + k^{n-1}|q'(0)|)}{(1+k^n)} \frac{(k-1)^n}{n} for\ n = 2, \end{aligned} \tag{25}$$

where $Q(z)=z^n\overline{P\left(\frac{1}{\overline{z}}\right)}$.

Proof of Lemma 4. Since $P(z)$ has all its zeros in $|z|\le k, k\ge 1$, we write

$$p(z)=a_n\prod_{j=1}^{n}\left(z-r_je^{i\theta_j}\right),$$

where $a_n\neq 0$ and $r_j\le k, j=1,2,\ldots,n$.
Then, clearly for points $e^{i\theta}, 0\le\theta<2\pi$, other than zeros of *P(z),* we have

$$\left|\frac{P(k^2e^{i\theta})}{P(e^{i\theta})}\right|=\prod_{j=1}^{n}\left|\frac{\left(k^2e^{i\theta}-r_je^{i\theta_j}\right)}{\left(e^{i\theta}-r_je^{i\theta_j}\right)}\right|$$

$$=\prod_{j=1}^{n}\left\{\frac{k^4+r_j^2-2k^2r_j\cos(\theta-\theta_j)}{1+r_j^2-2r_j\cos(\theta-\theta_j)}\right\}$$

$$\ge\prod_{j=1}^{n}k=k^n.$$

This implies,

$$|P(k^2e^{i\theta})|\ge k^n|P(e^{i\theta})|$$

for points $e^{i\theta}, 0\le\theta<2\pi$, other than the zeros of $P(z)$. Since this inequality is trivial for points $e^{i\theta}$, which are the zeros of $P(z)$, it follows that

$$|P(k^2z)|\ge k^n|P(z)| \; for \; |z|=1. \tag{26}$$

Let $G(z)$ = $P(kz)$ and $H(z)=z^n\overline{G\left(\frac{1}{\overline{z}}\right)}=z^n\overline{P\left(\frac{k}{\overline{z}}\right)}$. Applying inequality (19) to the polynomial *G(z),* we get for $|z|=1$ and $k\ge 1$,

$$|G(kz)|+|H(kz)|\le(k^n+1)Max_{|z|=1}|G(z)|$$
$$-(|G'(0)|+|H'(0)|)\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right) for \, n>2.$$

Equivalently, for $|z|=1$

$$|P(k^2z)|+k^n|P(z)|\le(1+k^n)Max_{|z|=k}|P(z)|$$
$$-(k|P'(0)|+k^{n-1}|Q'(0)|)\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right) for \; n>2.$$

This gives with the help of (26) for $|z| = 1$

$$2k^n |P(z)| \leq (1+k^n) Max_{|z|=k} |P(z)| \\ -(k|P'(0)|+k^{n-1}|Q'(0)|)\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right),$$

From which inequality (24) follows for $n > 2$.

The proof of the lemma for $n = 2$ follows on the same lines as that for $n > 2$, but using inequality (20) instead of inequality (19).

Next we prove the following result which gives an improvement of the above lemma by involving the term $\min_{|z|=k} |P(z)|$ as follows.

Lemma 5. If $P(z)$ is a polynomial of degree $n \geq 2$ having all its zeros in $|z| < k$, $k > 1$, and $P(0) \neq 0$, then

$$Max_{|z|=k} |P(z)| \geq \frac{2k^n}{1+k^n} Max_{|z|=1} |P(z)| + \left(\frac{k^n-1}{k^n+1}\right) Min_{|z|=k} |P(z)| \\ + \frac{(k|P'(0)|+k^{n-1}|Q'(0)|)}{(1+k^n)} \\ \times \left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right) for\, n > 2, \tag{27}$$

and

$$Max_{|z|=k} |P(z)| \geq \frac{2k^n}{1+k^n} Max_{|z|=1} |P(z)| + \left(\frac{k^n-1}{k^n+1}\right) Min_{|z|=k} |P(z)| \\ + \frac{(k|P'(0)|+k^{n-1}|Q'(0)|)}{(1+k^n)} \frac{(k-1)^n}{n} for\, n = 2, \tag{28}$$

where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$.

Proof of Lemma 5. Since $P(z)$ has all of its zeros in $|z| \leq k, k \geq 1$, then by Rouche's theorem, the polynomial $F(z) = P(z) + \lambda m$ with $|\lambda| \leq 1$, where $m = Min_{|z|=k} |P(z)|$, also has all its zeros in $|z| \leq k, k \geq 1$. So, on applying Lemma 4 for $n > 2$ to the polynomial $F(z)$, we have

$$Max_{|z|=k} |F(z)| \geq \frac{2k^n}{1+k^n} Max_{|z|=1} |F(z)| \\ + \frac{(k|F'(0)|+k^{n-1}|T'(0)|)}{\left(1+k^n\right)}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right),$$

Where $T(z)=z^n\,\overline{F\left(\frac{1}{\bar{z}}\right)}=z^n\left(\overline{P\left(\frac{1}{\bar{z}}\right)+\lambda m}\right)=Q(z)+\bar{\lambda}mz^n$

This implies for $n>2$

$$Max_{|z|=k}|P(z)|+|\lambda|m\geq\frac{2k^n}{1+k^n}Max_{|z|=1}|P(z)+\lambda m|+\frac{(k|P'(0)|+k^{n-1}|Q'(0)|)}{(1+k^n)}\times\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right),\tag{29}$$

If we choose z_0 so that $Max_{|z|=1}|P(z)|=|P(z_0)|$ and also choose argument of λ suitably, such that

$$Max_{|z|=1}|P(z)+\lambda m|=|P(z_0)|+|\lambda|m,$$

the inequality (29) becomes

$$Max_{|z|=k}\left|P(z)\right|+|\lambda|m\geq\frac{2k^n}{1+k^n}\left\{\left|P(z_0)\right|+|\lambda|m\right\}+\frac{\left(k\left|P'(0)\right|+k^{n-1}\left|Q'(0)\right|\right)}{\left(1+k^n\right)}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right),$$

Which on simplification gives,

$$Max_{|z|=k}|P(z)|\geq\frac{2k^n}{1+k^n}Max_{|z|=1}|P(z)|+|\lambda|\frac{(k^n-1)}{(k^n+1)}m+\frac{(k|P'(0)|+k^{n-1}|Q'(0)|)}{(1+k^n)}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right).$$

Finally letting $|\lambda|\to1$, we get the desired result for $n>2$.

The proof of inequality (28) follows on the same lines as the proof of (27), but instead of inequality(24) of Lemma 4, we use the inequality (25) of the same lemma.

3. Proof of the Theorem

The polynomial $G(z)=P(kz)=a_n\prod_{j=1}^{n}(kz-z_j)$ has all its zeros in $|z|\leq1$ and we have

$$\frac{G'(z)}{G(z)} = \sum_{j=1}^{n} \frac{1}{z - \left(\frac{z_j}{k}\right)}$$

so that for $0 \leq \theta < 2\pi$

$$\left|\frac{G'(e^{i\theta})}{G(e^{i\theta})}\right| \geq \mathrm{Re}\frac{e^{i\theta}G'(e^{i\theta})}{G(e^{i\theta})} = \sum_{j=1}^{n} \mathrm{Re}\frac{e^{i\theta}}{e^{i\theta} - \left(\frac{z_j}{k}\right)} \geq \sum_{j=1}^{n} \frac{k}{k+|z_j|}$$

This implies

$$Max_{|z|=1}|G'(z)| \geq \sum_{j=1}^{n} \frac{k}{k+|z_j|} Max_{|z|=1}|G(z)|,$$

equivalently

$$kMax_{|z|=1}|P'(kz)| \geq \sum_{j=1}^{n} \frac{k}{k+|z_j|} Max_{|z|=1}|P(kz)|. \tag{30}$$

Since $P'(z)$ is a polynomial of degree n — 1, it follows by (17) that

$$k\left(k^{n-1}Max_{|z|=1}|P'(z)| - (k^{n-1} - k^{n-3}\right)|P'(0)|) \geq \sum_{j=1}^{n} \frac{k}{k+|z_j|} Max_{|z|=k}|P(z)|.$$

Now using inequality (27) of Lemma 5 in above inequality, we get

$$\begin{aligned} k^n Max_{|z|=1}|P'(z)| - (k^n - k^{n-2}|P'(0)| \\ \geq \sum_{j=1}^{n} \frac{k}{k+|z_j|}\Bigg\{ \frac{2k^n}{1+k^n} Max_{|z|=1}|P(z)| \\ + \left(\frac{k^n - 1}{1+k^n}\right) \min_{|z|=k}|P(z)| \\ + \frac{(k|P'(0)| + k^{n-1}|Q'(0)|)}{(1+k^n)} \\ \times \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n-2}\right)\Bigg\} \end{aligned}$$

implying

$$
\begin{aligned}
Max_{|z|=1}|P'(z)| \geq \sum_{j=1}^{n}\frac{k}{k+|z_j|}\Bigg\{ & \frac{2}{1+k^n}Max_{|z|=1}|P(z)| \\
& +\frac{(k^n-1)}{k^n(1+k^n)}\min_{|z|=k}|P(z)| \\
& +\frac{(k|P'(0)|+k^{n-1}|Q'(0)|)}{k^n(1+k^n)}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)\Bigg\} \\
& +\left(1-\frac{1}{k^2}\right)|P'(0)|
\end{aligned}
$$

from which the theorem follows for $n > 2$.

The proof of the theorem for $n = 2$ follows on the same lines as that of $n > 2$, but instead of using inequalities (17) and(27) of Lemma 2 and Lemma 5 respectively, we use inequalities (18) and (28) of the respective lemmas.

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