DOI: 10.15415/mjis.2015.32015

Unique Fixed Point Theorem for Weakly Inward Contractions and Fixed Point Curve in 2-Banach Space

P RIYAS[1] AND KT RAVINDRAN[2]

[1]Department of Mathematics, K M M Govt. Women's College, Kannur, India-670004,

[2]P G Department of Mathematics, Payyannur College, Kannur

Email:riyasmankadavu@gmail.com

Received: October 10, 2014| Revised: December 1, 2014| Accepted: March 4, 2014

Published online: March 30, 2015

Abstract: In this paper, we prove unique fixed point theorem for weakly inward contractions on 2-Banach space and introduce the concept of fixed point curve in 2-Banach space.
2010 *AMS Classication*: 41A65, 46B20,47H10, 54H25.

Keywords: 2-normed space, 2-Banach Space, fixed point, weakly inward contraction, fixed point curve.

1 INTRODUCTION

The concept of 2-Banach space was investigated by S. Gähler and K Iseki]. This space was subsequently been studied by many authors. In 1988, Sam B Nadler and K Ushijima[9] introduced the concept of fixed point curves for linear interpolations of contraction mappings using Banach contraction principle. Recently, we have presented some interesting work on weakly inward contractions in normed space[5] which helps to define the concept of fixed point curve for weakly inward contractions in normed space. In this paper, we investigate such results for weakly inward contractions in 2-Banach Space. In [7] S Gähler introduced the following definition of a 2-normed space.

2 PRELIMINARIES

Definition 2.1[7]. Let X be a real linear space of dimension greater than 1. Suppose $\|\,,\,\|$ is a real valued function on $X \times X$ satisfying the following conditions:

Mathematical Journal of Interdisciplinary Sciences
Vol. 3, No. 2,
March 2015
pp. 173–182

1. $\|x, y\| = 0$ if and only if x and y are linearly dependent
2. $\|x, y\| = \|y, x\|$
3. $\|\alpha x, y\| = |\alpha|\|x, y\|$
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$

Then $\| \, , \|$ is called a 2-norm on X and the pair $(X, \| \, , \|)$ is called a 2-normed space. Some of the basic properties of 2-norms, that they are non-negative and $\|x, y + x\| = \|x, y\|$, $\forall x, y \in X$ and $\forall \alpha \in R$

Definition 2.2[7].: A sequence $\{x_n\}$ in a 2-normed space X is said to be convergent if there exists an element $x \in X$ such that

$$\lim_{n\to\infty} \| \{x_n - x\}, y \| = 0$$

for all $y \in X$.

Definition 2.3[7].: A sequence $\{x_n\}$ in a 2-normed space X is said to be a Cauchy sequence if for every $z \in X$, $\lim_{m,n\to\infty} \| x_n - x_m, z \| = 0$.

Definition 2.4[8]: A linear 2-normed space is said to be 2-Banach Space if every Cauchy sequence is convergent to an element of X.

Definition 2.5 [4]. Let X and Y be two 2-normed spaces and $T : X \to Y$ be a linear operator. The operator T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ of X converging to x, we have $T(\{x_n\}) \to T(x)$.

Definition 2.6 [4]. The closure of a subset E of a 2-normed space X is denoted by $\bar{E}$ and defined by the set of all $x \in X$ such that there is a sequence $\{x_n\}$ of E converging to x. We say that E is closed if $E = \bar{E}$.

Definition 2.7[6]. Let X be a linear space. A subset C of X is called convex(resp. absolutely convex) if $\alpha C + \beta C \subseteq C$ for every ; $\alpha,\beta > 0$(resp. $\alpha, \beta \in K$) with $\alpha + \beta = 1$(resp. $|\alpha| + |\beta| \leq 1$)

Denition 2.8 [4]. Let X be a 2-normed space. Then an operator T on X is said to be a contraction on X if for each $x, y \in X$ there exist some $k \in [0; 1)$ such that

$$\| Tx - Ty, z \| \leq k \| x - y, z \|, \ \forall z \in X.$$

Definition 2.9 [6]. Let C be a non-empty subset of a 2-Banach space X. Then a mapping $T : C \to X$ is said to be inward mapping(respectively weakly inward) if $Tx \in I_C(x)$ (respectively $Tx \in \overline{I_C(x)}$) for $x \in C$ where

$$I_C(x) = \{(1-t)x + ty : y \in C \, and \, t \geq 0\}$$

3 MAIN RESULTS

Proposition [3.1]: Let X be a 2-Banach space. Suppose that there exist a sequence $\{x_n\}$ in X such that

$$\| x_n - x_{n+1}, z \| \le \phi(x_n) - \phi(x_{n+1}) \; for \; n = 0,1,2\ldots$$

where $\phi : X \to [0,\infty)$ is any function. Then the sequence $\{x_n\}$ converges to some point $v \in X$.

Proof: For any $z \in X$,

$$0 \le \| x_n - x_{n+1}, z \| \le \phi(x_n) - \phi(x_{n+1}) \;\; for \;\; n = 0,1,2\ldots$$

This shows that the sequence $\{\phi(x_i)\}$ is decreasing and bounded below by zero.

Now, for any m = 0, 1, 2... we have

$$\sum_{n=0}^{m} \| x_n - x_{n+1,} z \| \le \| x_0 - x_1, z \| + \| x_1 - x_2, z \| + \ldots + \| x_m - x_{m+1}, z \|$$
$$\le \phi(x_0) - \phi(x_1) + \phi(x_1) - \phi(x_2) + \ldots + \phi(x_m) - \phi(x_{m+1})$$
$$= \phi(x_0) - \phi(x_{m+1})$$
$$\le \phi(x_0) - \inf_n \phi(x_n).$$

Letting $m \to \infty$, we have $\sum_{n=0}^{\infty} \| x_n - x_{n+1}, z \| < \infty$.

This implies that $\| x_n - x_{n+1}, z \| \to 0 \;\; as \;\; n \to \infty$.

Thus, for any positive integer m and n with $m > n$ and for all $z \in X$,

$$\| x_n - x_m, z \| \le \| x_n - x_{n+1}, z \| + \| x_{n+1} - x_{n+2}, z \| + \ldots \| x_{m-1} - x_m, z \|$$
$$\to 0 \; as \; n \to \infty$$

This shows that $\{x_n\}$ is a Cauchy sequence in a 2-Banach space X. Therefore there exist some point $v \in X$ such that $x_n \to v$ *as* $n \to \infty$.

Theorem [3.2]: Let X be a 2-Banach space and T be a continuous self map on X. Suppose that there exist a function $\phi: X \to [0,\infty)$ such that

$$\| x - Tx, z \| \le \phi(x) - \phi(Tx), \;\; \forall x, z \in X.$$

Then T has a fixed point in X.

Proof: Let $x_0 \in X$ and let $\{x_n\}$ be a sequence in X dened by Picard iteration $x_n = T^n(x_0)$ for n = 0, 1, 2...

Then for any *n*,

$$\|x_n - x_{n+1}, z\| = \|x_n - T(x_n), z\| \\ \leq \phi(x_n) - \phi(T(x_n)) \\ = \phi(x_n) - \phi(x_{n+1})$$

By Proposition[3.1], there exist some $v \in X$ such that $x_n \to v$ as $n \to \infty$. Since T is continuous, $T(x_n) \to T(x)$ as $n \to \infty$. Now, for any $z \in X$,

$$\|v - Tv, z\| \leq \|v - x_n, z\| + \|x_n - Tv, z\| \\ = \|v - x_n, z\| + \|T(x_n) - Tv, z\| \\ \to 0 \text{ as } n \to \infty$$

This implies that there exist a point $v \in X$ such that $Tv = v$.

Theorem [3.3]: Let X be a 2-Banach space and $\phi : X \to [0,1)$ be continuous on X. Suppose that for each $u \in X$ with inf $\inf_{x \in X} \phi(x) \neq \phi(u)$, there exist $v \in X$ such that $u \neq v$ and

$$\|u - v, z\| \leq \phi(u) - \phi(v), \ \forall \ z \in X.$$

Then there exist an $x_0 \in X$ such that $\phi(x_0) = \inf_{x \in X} \phi(x)$.

Proof: Suppose that $\inf_{x \in X} \phi(x) \neq \phi(y)$ for all $y \in X$ .. Fix an element $u_0 \in X$ such that $\inf_{x \in X} \phi(x) < \phi(u_0)$. By our assumption there exist an element $u_1 \in X$ such that $u_1 \neq u_0$ and

$$\|u_0 - u_1, z\| \leq \phi(u_0) - \phi(u_1), \ \ \forall z \in X.$$

Proceeding inductively choose $u_{n-1} \in X$.

Let

$$S_n = \left\{x \in X : \ \|u_{n-1} - x, z\| \leq \phi(u_{n-1}) - \phi(x), \ \forall z \in X\right\}.$$

Choose $u_n \in S_n$ such that $\phi(u_n) \leq \frac{1}{2}[\phi(u_{n-1}) + \inf_{x \in S_n} \phi(x)]$....................(1)

Because $u_n \in S_n$, we get

$$\|u_{n-1} - u_n, z\| \leq \phi(u_{n-1}) - \phi(u_n), \ \forall z \in X \ \text{and } n.$$

By Proposition [3.1], it follows that there exist $u \in X$ such that $u_n \to u$ and

$$\| u_{n-1} - u, z \| \leq \phi(u_{n-1}) - \phi(u), \ \ \forall z \in X \ and \ n.$$

By our assumption, there exist a $v \in X$ such that $u \neq v$ and

$$\| u - v, z \| \leq \phi(u) - \phi(v) \ \ \forall z \in X........(2)$$

Therefore,

$$\begin{aligned} \phi(v) &\leq \phi(u) - \| u - v, z \| \\ &\leq \phi(u) - \| u - v, z \| + \phi(u_{n-1}) - \phi(u) - \| u_{n-1} - u, z \| \\ &= \phi(u_{n-1}) - [\| u_{n-1} - u, z \| + \| u - v, z \|] \\ &\leq \phi(u_{n-1}) - \| u_{n-1} - v, z \| \ \forall z \in X. \end{aligned}$$

This imples that $v \in S_n$. It follows from (1) and (2) that

$$\phi(v) < \phi(u) = \lim_{n \to \infty} \phi(u_n) \leq \phi(v).$$

Which is a contradiction.
Hence there a point $x_0 \in X$ such that $\phi(x_0) = \inf_{x \in X} \phi(x)$.

Theorem [3.4]: Let X be a 2-Banach space and T be a self map on X. Suppose that there exist a continuous function $\phi : X \to [0, \infty)$ such that

$$\| x - Tx, z \| \leq \phi(x) - \phi(Tx), \ \forall \ x, y, z \in X.$$

Then T has a xed point in X.

Proof: Let $u \in X$ be fixed and define

$$C = \{ x \in X : \| u - x, z \| \leq \phi(u) - \phi(x) \ \ \forall z \in X \}$$

Then clearly C is non-empty. If $x \in \overline{C}$, then there exist a sequence $\{x_n\}$ in C such that $x_n \to x$ and

$$\| u - x_{n,} z \| \leq \phi(u) - \phi(x_n) \ \ \forall z \in X$$

Therefore by the continuity of ϕ, it follows that $\| u-x,z \| \le \phi(u)-\phi(x) \ \ \forall z \in X$. This shows that *C* is a non-empty closed subset of *X*. We now show that *C* is invariant under *T*.

For each $x \in C$, we have $\| u-x,z \| \le \phi(u)-\phi(x) \ \forall z \in X$ and therefore

$$\begin{aligned} \phi(Tx) &\le \phi(x) - \| x-Tx,z \| \\ &\le \phi(x) - \| x-Tx,z \| + \phi(u) - \phi(x) - \| u-x,z \| \\ &= \phi(u) - [\| x-Tx,z \| + \| u-x,z \|] \\ &\le \phi(u) - \| u-Tx,z \|, \ \ \forall z \in X \end{aligned}$$

It follows that $Tx \in C$ and hence *C* is invariant under *T*.
Now, suppose that $x \ne Tx$ for all $x \in C$. Then, for each $x \in C$, there exist $y \in C$ such that $x \ne y$ and

$$\begin{aligned} \| x-y,z \| &\le \| x-u,z \| + \| u-y,z \| \\ &\le \phi(x)-\phi(u)+\phi(u)-\phi(y) \\ &= \phi(x)-\phi(y), \ \ \forall z \in X. \end{aligned}$$

Since ϕ is continuous on a closed subset *C* of *X*, by theorem[3.3], there exist some $x_0 \in C$ such that $\phi(x_0) = \inf_{x \in C} \phi(x)$. Hence for such an x_0 and for some $z_0 \in X$ we have

$$\begin{aligned} 0 < \| x_0 - Tx_0, z_0 \| &\le \phi(x_0) - \phi(Tx_0) \\ &\le \phi(Tx_0) - \phi(Tx_0) \\ &= 0. \end{aligned}$$

This is a contradiction. Hence there exist some point $v \in C$ such that $Tv = v$.
Proposition [3.5]: Let *C* be a non empty closed convex subset of a 2-Banach space *X*. Then $T : C \to X$ is weakly inward if and only if

$$\lim_{h \to 0+} \frac{d((1-h)x + hTx, C)}{h} = 0, \ \ \forall x \in C,$$

where $d(x,C) = \inf \{ \| x-y,z \| : y \in C \ \ and \ \ z \in X \}$

Proof: Suppose that

$$\lim_{h\to 0+}\frac{d((1-h)x+hTx,C)}{h}=0,\ \ \forall x\in C$$

Let $x\in C$ and choose $t\in[0,1)$. Then for any $\varepsilon>0$, we can find $y\in C$ such that

$$\|(1-t)x+tTx-y,z\|< d((1-t)x+tTx,C)+t\varepsilon,\ \forall z\in X.$$

$$\Rightarrow\|Tx-[(1-t^{-1})x+t^{-1}y],z\|<\frac{d((1-t)x+tTx,C)}{t}+\varepsilon,\ \forall z\in X.$$

If we choose $t=1,\frac{1}{2},\frac{1}{3},..,$ then there exist a sequence $x_n=(1-t_n^{-1})x+t_n^{-1}y\in I_C(x)$ such that $\|Tx-x_n,z\|<\frac{d((1-t_n)x+t_nTx,C)}{t_n}+\varepsilon, \forall z\in X.$ Letting $n\to\infty$, we obtain $\lim_{n\to\infty}\|Tx-x_n,z\|=0,\ \forall\ z\in X$. Thus there exist a sequence $\{x_n\}\in I_C(x)$ such that $x_n\to Tx$. It follows that $Tx\in\overline{I_C(x)}$. Hence T is weakly inward.

Conversely suppose that T is weakly inward. That is $Tx\in\overline{I_C(x)},\ \forall x\in C$. Then for any $\varepsilon>0$, there exist $y\in I_C(x)$ such that

$$\|y-Tx,z\|<\varepsilon,\ \forall z\in X.$$

Since C is convex, there exist some $h_0>0$ such that $(1-h)x+hy\in C$ for $0<h\le h_0$. For these h, we have

$$\begin{aligned}\frac{d((1-h)x+hTx,C)}{h}&\le\frac{\|(1-h)x+hTx-((1-h)x+hy),z\|}{h}\\&=\frac{\|hTx-hy,z\|}{h}\\&=\|Tx-y,z\|\\&<\varepsilon,\forall\varepsilon>0\ \ and\ 0<h\le h_0.\end{aligned}$$

Therefore, $\lim_{h\to 0+}\frac{d((1-h)x+hTx,C)}{h}=0,\ \forall x\in C$

Theorem [3.6]: Let C be a non empty closed convex subset of a 2-Banach space X and $T: C \to X$, a weakly inward contraction mapping. Then T has a unique fixed point in C.

Proof: Let $k \in (0,1)$ be such that

$\| Tx - Ty, z \| \le k \| x - y, z \|$, $\forall x, y \in C$ *and* $z \in X$. Choose $\varepsilon > 0$, so small that $k < \frac{1-\varepsilon}{1+\varepsilon}$. By proposition [3.4], $\lim_{h \to 0+} \frac{d((1-h)x + hTx, C)}{h} = 0$, $\forall x \in C$. Then for any $x \in C$ with $x \neq Tx$ and for every z with $z \notin span\{x - Tx\}$ there exist $h \in (0,1)$ such that

$$d((1-h)x + hTx, C) < h\varepsilon \| x - Tx, z \|$$

By the denition of distance, there exist some $y \in C$ such that

$$\| (1-h)x + hTx - y,\ a \| < h\varepsilon \| x - Tx, z \|,\ \forall a \in X \text{ and } z \notin span\{x - Tx\} \ldots (1)$$

$$\begin{aligned} \Rightarrow h\varepsilon \| x - Tx, z \| &> \| x - y - h(x - Tx), a \| \\ &\ge \| x - y, a \| - h \| x - Tx, a \| \end{aligned}$$

Therefore,

$$\| x - y, a \| < h[\varepsilon \| x - Tx, z \| + \| x - Tx, a \|],\ \forall a \in X \text{ and } z \notin span\{x - Tx\}$$

which implies that

$$\| x - y, z \| < h(1+\varepsilon) \| x - Tx, z \|,\ \forall z \notin span\{x - Tx\} \ldots (2)$$

Using (1) and (2),

$$\begin{aligned} \| y - Ty, z \| \le\ & \| y - ((1-h)x + hTx), z \| \\ & + \| (1-h)x + hTx - Tx, z \| + \| Tx - Ty, z \| \\ & < h\varepsilon \| x - Tx, z \| + (1-h) \| x - Tx, z \| + k \| x - y, z \| \\ & = \| x - Tx, z \| + (\varepsilon - 1)h \| x - Tx, z \| + \frac{1-\varepsilon}{1+\varepsilon} \| x - y, z \| \\ & - \left(\frac{1-\varepsilon}{1+\varepsilon} - k \right) \| x - y, z \| \\ & < \| x - Tx, z \| - \left(\frac{1-\varepsilon}{1+\varepsilon} - k \right) \| x - y, z \| \ldots\ldots\ldots (3) \end{aligned}$$

For any $x \in C$ with $x \neq Tx$, there is a $y \in C$, denote it by $f(x)$ where f is a self mapping on C. Then by putting $\phi(x) = \left(\frac{1-\varepsilon}{1+\varepsilon} - k\right)^{-1} \| x - Tx, z \|$, we obtain a continuous map $\phi : C \to [0,\infty)$ such that

$$\| x - f(x), z \| < \phi(x) - \phi(f((x))),\ \forall z \notin span\{x - Tx\} \ldots\ldots(4)$$

Then by theorm[3.4], f has a fixed point in C which contradicts the strict inequality(4). Hence there exist some $v \in C$ such that $Tv = v$. Now assme that v and w are any two fixed points of T. Then we have

$$\begin{aligned} \| v - w, z \| &= \| Tv - Tw, z \| \\ &\leq k \| v - w, z \|,\ \forall z. \\ \Rightarrow (1-k) \| v - w, z \| &\leq 0, \forall z \ and\ 0 < k < 1 \\ \Rightarrow \| v - w, z \| &\leq 0, \forall z \\ \Rightarrow v &= w. \end{aligned}$$

Hence T has a unique fixed point in C.

Theorem [3.7][5]: Let X be a 2-Banach space, C be a non-empty closed and convex subset of X and let $T_1, T_2 : C \to X$, be two weakly inward contractions. Then there exists a family of weakly inward contraction maps $\{\phi_t : C \to X : t \in [0,1]\}$ such that, $\forall t \in [0,1], \phi_t$ has a unique fixed point x_t in C.

Proof: Proof is similar to that in [5].

Denition [3.8](Fixed point curve): Let T_1 and T_2 be two weakly inward contraction on a closed, convex and bounded subset C of a 2-Banach space X and $F_W(T_1, T_2)$ denote the set of all fixed points of $\phi_t, t \in [0,1]$. Then the map $G : [0,1] \to F_W(T_1, T_2)$ defined by $G(t) = x_t$ is called the *fixed point curve* for T_1 and T_2.

Lemma [3.9][5]: Let $T_1, T_2 : C \to X$ be two weakly inward contractions on a closed, convex and bounded subset C of a 2-Banach space X. Suppose that T_1 and T_2 have no common fixed points. Then the map $G : [0,1] \to F_W(T_1, T_2)$ defined by $G(t) = x_t$ is one to one.

Corollary [3.10]: The fixed point curve $G : [0,1] \to F_W(T_1, T_2)$ is continuous.

Corollary [3.11]: $F_W(T_1, T_2)$ is a closed set.

REFERENCES

[1] F Raymond and Y J Cho, *Geometry of Linear 2-Normed Space*, Nova Science Publishers, Inc. Hauppauge, Ny(2001).

[2] K Iseki, *Fixed point theorems in 2-Banach Spaces*, Math Seminar Notes, Kobe Univ, **2**(1976),11-13.

[3] Mehmet Kir and Hukmi Kiziltunc, *Some New Fixed Point Theorems in 2-Normed Space*, Int. Journal of Math. Analysis. Vol.**7**(2013).

[4] P K Harikrishnan and K T Ravindran, *Some Properties of Accretive operators in Linear 2-Normed Spaces*, International Mathematical Forum, vol.**6**, (2011) no.59, 2941-2947.

[5] P. Riyas and K. T. Ravindran, *Fixed Point Curve for Weakly Inward Contractions and Approximate Fixed Point Property, Anal*. Theory Appl., **29** (2013), pp. 342-347.

[6] R P Agarwal, Donal O Regan, D R Sahu *Fixed Point Theory for Lip- schitzian Type Mappings with Applications* Springer Science, Business Media, LLC(2009).

[7] S. Gähler, *Lineare 2- normierte Rume*, Math.Nachr.28(1964)1-43.

[8] S. Gähler, Uber 2-Banach räume, Math. Nachr., **42**(1969), 335-347. http://dx.doi.org/10.1002/mana.19690420402

[9] Sam B Nadler, K Ushijima, *Fixed Point Curves for Linear Interpolations of Contraction Maps*, Applicable Analysis, Vol-**29**,1988.

[10] Siddiqi, A.H., 2-normed spaces, Aligarh Bull. math., (1980), 53-70.

[11] T. Vidali, *Fixed Point Curve Generated by Nonexpansive Mappings*, Fixed Point Theory, Volume **6**, No 2.2205,333-339.