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The Emergence of the Idea of Irrationality In Renaissance Theoretical Music Contexts

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Abstract: This paper intends to consider the interrelationships between mathematics and music in the context of changes in the conception of music during the Renaissance. Western music developed from relying on a cosmological-mathematical-speculative model, in which attention was focused on the rational activity of speculation, to a mathematical-empirical model, in which the main emphasis lay on the quality of the sound itself and on its laws and effects on people. Musical concepts such as temperament, division of the tone, changes in the foundations of theoretical music, mathematical-structural changes in theories of ratio and consequently the emergence of the idea of irrationals and number continuum in theoretical music contexts will be considered here in order to understand the substratum of such a change in the conception of western music.

Keywords: maths/music, Renaissance, aritmetization of ratios, numbers as continuum, irrationals in theoretical music

1. BACKGROUND

In the experiment of the monochord, Pythagoras is credited with having established the correspondence between musical intervals and ratios of a string, discovering that certain intervals could be produced by dividing the string in simple ratios $a:b$ such that b represented the whole string whereas a represented a part of the string. In particular, the intervals of the octave, fifth and fourth were produced by simple ratios 1:2; 2:3 and 3:4, respectively. These intervals were called perfect consonances, and the Pythagorean consonances consisted strictly of the intervals whose underlying ratios were formed only by elements of the *Tetraktys*, the series of numbers 1,2,3,4, whose sum results in 10 [118,4].

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In the context of Pythagoras' experiment, if now one considers – on the one hand, that a fourth, which is produced by 3:4, composed to a fifth, which is produced by 2:3, results musically in an octave, which is produced by 1:2; and on the other hand, that such an operation thus corresponds mathematically to taking 3:4 of the string followed by taking 2:3 of the remainder, which means taking (2:3)(3:4) of the string, i.e. (1:2) of the string – Pythagoras' experiment seems to tell us more than the very general point that mathematical ratios underlie musical intervals. More specifically, it tells us that the compounding ratios underlie the composition of musical intervals, and possibly due to this link, even that composition of ratios in a Euclidean fashion is handled in this way.

In Greece, Pythagoras' experiment represents the beginning of a science oriented toward mathematics. It shows how a mathematical order is inherent in the physical space, thus corroborating such an order as the origin and foundation of harmony. Pythagoras' discovery concerning the monochord, both in Greece and after the Hellenistic age, casts light on a large number of discussions about musical theory that have ratios as their basis. This feature shaped the conception of western music with a cosmological-mathematical-speculative understanding until the Renaissance, when a mathematical-empirical one began to emerge.

Among the factors involved in such a change in the conception of western music, the advent of polyphony played an important part. It implied the need of non-integers in the ratios underlying musical intervals, and thus a structural change in the numerical system over which the musical scales were developed. Such a change culminated in the systematization of the equal temperament. In this case, most of the concords are made slightly smaller or bigger in the tuning of the scale, so that none are left distasteful for the sake of execution of polyphonic music. In the context of theoretical music, such a structural change brought about the need for the division of the tone and in particular the need for the division of a ratio in equal parts – a division that would bring out the limitations and stiffness of a Pythagorean musical model that involved the search for a perfect system of intonation based on ratios between commensurable magnitudes. These changes would also eventually bring into question the rigid Pythagorean distinction between consonance and dissonance, defined by the first four numbers.

At the end of the fifteenth century and beginning of the sixteenth, in mathematical contexts, changes in the conception of ratio brought about the strengthening of the arithmetical theory of ratios, contributing in a wider sense to the arithmetization of the theories of ratio and at the same time to the use of geometry instead of Pythagorean arithmetic as the instrument for solving structural problems in theoretical music.

1.1 The construction of the Pythagorean scale and its implications

Boethius' conception of music had great influence in medieval music theory, which gave emphasis to mathematical-metaphysical-cosmological speculation. In the Middle Ages, *ars musica* was part of the *Quadrivium*, the mathematical sciences of the seven liberal arts. Music was a pure science. Practice and the quotidian had no place in the system of the seven liberal arts.

Only in the late Middle Ages, the reality concerning perceptible proprieties of music once again developed: music became aimed at the empirical experiences and at practice. According to Dickreiter, this development was caused in the thirteenth century by the influence of the Arabic musical theory of Al-Farabi, mediated by Gundilissalinus [174,10]. From this period on, the empirical conception of music gained growing importance, reaching its peak at the end of the fifteenth century.

In the context of such developments, the limitations and stiffness of the Pythagorean musical model became obvious, namely, the strict distinction between dissonance and consonance – defined by the first four natural numbers – and the composition of intervals determined by ratios of these numbers. The main source of the music theory of the early Pythagoreans was the fact that the ratios 1:2, 2:3 and 3:4 determine the perfect Greek consonances of the octave, fifth and fourth, respectively, and also, as a consequence, that the composition of contiguous musical intervals is determined by the compounding ratios of the respective intervals using the classical Greek method mentioned above in the discussion of its application by Euclid.

The Pythagoreans established a mathematical construction of the musical scale making use of these three Greek consonances employing the compounding of ratios. For instance, in order to compound the fifth with the fifth, that is, to compound 2:3 with 2:3, it is required to find a:b and b:c such that a:b is proportional to the first 2:3, and b:c is proportional to the second 2:3; a condition which in the present case is met, for instance, by the ratios 4:6 for a:b and 6:9 for b:c, both of them being proportional to 2:3, resulting in the compound ratio 4:9. Accordingly, to decompound an octave 1:2 from 4:9 means to find a: b and c: b so that a:b is proportional to the first 1:2 and c:b is proportional to the second 4:9. This condition is met, for instance, by the ratios 9:18 for a:b and 8:18 for c:d, resulting in the ratio 8:9, which is a *second*.

In order to build the Pythagorean scale, let us consider the notes produced by a cycle of *octaves* and by a cycle of *fifths*, meaning the notes produced by the ratios 1:2, 1:4, 1:8 etc and 2:3, 4:9, 8:27 etc, respectively. There is no loss of generality if one considers *do* as the initial note. Now one tries to determine each note produced by the cycle of *fifths* relative to the note in the cycle of the *octaves* that is immediately below. For instance, after two *fifths* one produces a note generated by the ratio 4:9. In this case, the *octave* which is immediately below is produced by the ratio 1:2, therefore, the note produced in the Pythagorean scale is generated by the ratio 8:9, which is a *re*. The next note generated by the cycle of *fifths* is produced by the ratio 8:27. The *octave* immediately below is still produced by the ratio 1:2, so the note produced in the Pythagorean scale is generated by the ratio 16:27, which is a *la.* The next note of the cycle of *fifths* is caused by the ratio 16:81. In this case, the *octave* which is immediately below is induced by the ratio 1:4, so the note produced in the Pythagorean scale is generated by the ratio 64:81, which is a *mi*, and so on. Such a procedure generates the Pythagorean scale from cycles of fifths and octaves.

This process can be obtained equivalently if one decompounds an *octave* whenever the note produced by the cycles of fifths goes beyond an *octave*. Since the fourth is an octave decompounded by a fifth or an octave decomposing a fifth, it is enough to use octaves and fifths to generate the Pythagorean scale. Thus, starting for instance from C, if one compounds a fifth with a fifth, which mathematically means to compound 2:3 with 2:3, one obtains a composed D, which corresponds to the ratio 4:9; after dropping an octave, which corresponds mathematically to decompound from it 1:2, the result is the note D with the ratio 8:9.

If now one compounds a second with a fifth, which means mathematically to compound 8:9 with 2:3, one obtain the note A corresponding to the ratio 16:27, and so on, obtaining the following diatonic scale, called the *Pythagorean scale:*

C	D	E	F	G	A	B	c
1:1	8:9	64:81	3:4	2:3	16:27	128:243	1:2

Continuing this process for the construction of the scale it provides for F# in the superior octave the ratio (128:243)(2:3), namely 256:729, thus 512:729 for F# in this octave, and so on, according to the table presented in the following.

The columns in the table 1 presented below show the ratios and notes, respectively, generated by compounding *fifths* followed by decompounding an octave whenever the generated note is higher than the octave, or mathematically, whenever the ratio is smaller than 1:2, so that one always keeps the notes in the same octave, or mathematically one keeps the ratios between 1:1 and 1:2.

In table 2, the lines of table 1 are reordered according to the sizes of the ratios, thus, producing the chromatic Pythagorean scale. The last column expresses the ratio expressed as a decimal underlying the interval between two consecutive notes showing two types of half tones in the Pythagorean scale.

Table 1: The construction of the Pythagorean scale

ratio	note
1:1	C
2:3	G
8:9	D
16:27	A
64:81	E
128:243	B
512:729	F#
2048:2187	C#
4096:6561	G#
16384:19683	D#
32768:59049	A#
131072:177147	F
262144:531441	c

Now one obtains the following chromatic scale:

C	C#	D	D#	E	F	F#	G	G#	A	A#	B	c
1:1	2048:2187	8:9	16384:19683	64:81	131072:177147	512:729	2:3	4096:6561	16:27	32768:59049	128:243	262144:531441

The chromatic Pythagorean scale is:

The notes F and C found in the table presented before, 131072:177147 = 0,73990527641 and 262144:531441 = 0,49327018427, respectively are slightly different from 3:4 and 1:2, which are those of the Pythagorean scale. As mentioned before, a *do* (C) was considered the initial note, so this result means that the *fourth* and the *octave* obtained by the afore mentioned process are not

Table 2: The construction of the chromatic Pythagorean scale

note	ratio	distance
C	1:1	
C#	2048:2187	0.936442615
D	8:9	0.94921875
D#	16384:19683	0.936442615
E	64:81	0.94921875
F	131072:177147	0.936442615
F#	512:729	0.94921875
G	2:3	0.94921875
G#	4096:6561	0.936442615
A	16:27	0.94921875
A#	32768:59049	0.936442615
B	128:243	0.94921875
c	262144:531441	0.936442615

C	C#	D	D#	E	F	F#	G	G#	A	A#	B	c
1:1	2048:2187	8:9	16384:19683	64:81	3:4	512:729	2:3	4096:6561	16:27	32768:59049	128:243	1:2

exactly the same obtained in the discovery with the monochord, attributed to Pythagoras, of the numerical ratios corresponding to the main intervals of the musical scale.

One can put it down to the fact that the afore mentioned cycles of the *octaves* and of *fifths* don't meet, as it will be shown later and that the best approximation for this meeting occurs with 7 *octaves* and 12 *fifths*. So it is plausible that the ratios determined by the *fourth* and by the *octave* in the Pythagorean scale result from the experiment of the monochord, namely, from the musical perception that such ratios produce these intervals, they were the perfect consonances in the context of Greek music and which were capable of being produced by ratios composed by small numbers, being easier to relate to the intervals using only the ear. It is confirmed by the fact that the notes F and

C are the last ones – considering *do* as the first note – produced by the cycles of *fifths* if one considers the approximation mentioned before of 7 *octaves* with 12 *fifths*, namely C, G, D, A, E, B, F#, C#, G#, D#, A#, "**F**", "**C**". Another factor somewhat related to the latter argument is that the scale was supposed to be built inside an *octave* – produced by 1:2 in the experiment of the monochord –, since the notes whose differences are an integer number of octaves show a semantic similarity, manifested for instance when a child tries to repeat a song sung by an adult, what the child does with a difference of octaves since he/she cannot reach the original notes emitted by such an adult. So the adjustment of the notes produced by the cycles of *fifths* to the first *octave* are made naturally using the octave obtained in the experiment of the monochord, that is, produced by the ratio 1:2. Using this fact, the fourth 3:4 could be considered from this experiment or if the cycle is generated by decompounding *fifths* and compounding an *octave* whenever the note produced falls outside of the initial *octave*, a *fourth* 3:4 could be produced by decompounding an *fifth* from the initial note followed by the composition of an *octave*, which will result in 1:1 decompounded by 2:3, which is 3:2 and this decompounded by 1:2 which is 3:4.

As mentioned previously, there is no loss of generality if one considers *do* as the first note in the sense that this process would result in the same dynamics if it had begun with another note. For instance, if one begins with F an octave lower – instead of C –, which corresponds to the ratio 3:4 decompounded by 1:2, that is, 3:2, then after compounding it with a *fifth*, generated by 2:3, it will result in 1:1, that is C, from which the process initiated. And the continuation of the process will happen as in the previously mentioned scale. If one begins with F (3:2), the cycle will finish with the F found in the table (131072:177147), whereas if the process begins with C (1:1), the cycle will finish in the second c found in the table (262144:531441). In the first case, the ratio between the last F and the one octave higher F, which is 3:4; is (131072x4):(177147x3), which is (524288:531441) = 0,98654037. In the second case, the ratio between the last c and the C one octave higher, which is 1:2; is (262144x2):(531441), that is also (524288:531441) = 0,98654037. This ratio is not 1:1, but (524288:531441) = 0,98654037, and therefore neither cycles meet, as one would expect when one thinks of the same process on a keyboard. Nevertheless, 524288:531441 = 0,98654037 is almost 1:1 and is called the *Pythagorean comma*, as will be shown in the following.

Anachronistically, the semitone between F and F# is produced by (512:729)/(3:4), that is 2048:2187, approximately 0.9364; whereas the semitone between G and F# is produced by (2:3)/(512:729), which is 1458:1536, approximately 0.9492. According to the table, one obtains a scale in which the tone is not equally divided, but almost equally, as is verifiable in the last column of the table, where each tone in such a scale is divided into two different, albeit quasi-equal, semitones.

In a general sense, such a process leads to an asymmetric scale, namely a scale in which not all equal intervals are produced by the same ratio. This is the price paid for constructing a scale by a completely pure method, namely based only on intervals produced by ratios of the *Tetraktys*. Such a compromise between symmetry and purity in the construction of a scale will be shuddered at with the advent of the equal temperament, when a conceptual change has occurred in the mathematical basis of the scale, now based on irrational numbers, as will be shown in the following.

2. TEMPERAMENT

The Pythagorean scale was based on the *Tetraktys.* This scale was predominant up until the late Middle Ages. J. Murray Barbour presents the just intonation of Ramos de Pareja from 1492 as the earliest proposal for replacing the Pythagorean system (Barbour 1953). During the thirteenth to the fifteenth centuries, the advancement of the motet contributed to the development of polyphony. In medieval polyphony only perfect intervals were regarded as true consonances. The perfect intervals or consonances were the *octave*, the *fifth*, the *fourth* and the *unison* or any compound of one of these intervals. The beginning of the *Ars Nova* in the fourteenth century brought about a change in musical attention from melody to counterpoint and harmony, as a consequence the introduction of imperfect consonances of thirds and sixths in the fifteenth century, when theorists began to accept such intervals as consonances, establishing a distinction between *concordatia perfecta* and *concordatia imperfecta.* Contrasting with the expression “perfect consonance”, “imperfect consonance” refers to major and minor thirds and sixths and their compounds (tenth, thirteenth etc). In the late Middle Ages, serious alternatives to Pythagorean tuning were first considered by musical theorists [445,7] in order to answer the need of a new musical language provided by polyphony.
Anachronistically speaking, this means that, supposing both these both cycles meet, there would be *m* and *n* integers such that $(2:3)^n = (1:2)^m$, that is, $3^n = 2^{m+n}$, which is impossible, since the left term is odd and the right is even. This implicit contradiction and maybe a defense-mechanism of nature against order caused the emergence of different solutions throughout history. The Pythagoreans would consider a point where there is a good approximation between the cycles mentioned above. In fact after repeating such a process 12 times, the 12 musical notes generated by such a process are those of the chromatic scale and the result is very close to 7 full octaves – if n = 12 and

m = 7, then 12 fifths exceed by a bit 7 octaves: the difference is precisely the *Pythagorean comma*. It is possible to visualize if we express the Pythagorean comma mathematically. Such a difference is given thus by $(2:3)^{12}/(1:2)^{7}$, which is $2^{19}/3^{12}$ = 524288: 531441, which is the *Pythagorean comma*. This interval is tolerable in homophonic contexts but great enough to cause problems in harmonic intervals in polyphonic contexts, for instance, if a fifth lacking a Pythagorean comma is executed with several musical lines simultaneously; and to unleash the need for a truly structural change in the mathematical structure underlying the standard and persistent medieval system of intonation predominant since Antiquity. This means that in the Pythagorean intonation, the tuning of the scale comprises eleven pure fifths and an impure fifth – made of a pure one minus the Pythagorean comma – called the wolf's fifth. This was a way of adjusting the cycles of fifths and the cycles of octaves, in an attempt to maintain purity.

Temperaments are necessary mainly because natural intervals do not adjust themselves in other natural intervals. For example, three major natural thirds do not comprise an octave for nearly 1/5 of a whole tone; four minor natural thirds exceed an octave by a bit; the cycles of natural fifths do not meet the cycles of octaves as was shown above; a major second obtained from the subtraction of a minor natural third from a natural fourth is smaller than that obtained from the subtraction of a natural fourth from a natural fifth, etc.

The oldest temperament known in western music was the Pythagorean with the majority of natural fifths. The alternatives to the Pythagorean tuning consisted in the adjustment of the cycles mentioned above in other ways, which would inevitably and eventually abstain from purity in favor of similarity in the intervals for the systematization of the equal temperament. The appearance of such alternatives to the Pythagorean tuning seems to have occurred for the first time in the late Middle Ages. The fourteenth century, for example, already witnessed attempts to establish other temperaments besides the Pythagorean and even what one could call a proposal of equal temperament. According to Ellsworth, it appears as one of the five treatises on music theory, found in an anonymous fourteenth-century manuscript, dated in Paris, 12 January, 1375, for what appears to be a highly practical system, based upon equal temperament [445,3].

Unequal temperaments, for instance with natural thirds, flourished in the late Middle Ages and the Renaissance. Such proposals were surpassed by equal temperament inasmuch as music became more chromatic and extended to all tonalities.

The equal temperament consists in equal distribution, as will be shown in the following, the Pythagorean comma among the twelve fifths of the cycles.

The equal temperament consists of adjusting twelve fifths inside of seven octaves. Starting from a length l_0, the composition of seven octaves is produced by $(1/2)^7 l_0 = 0.0078125\ldots$whereas the composition of twelve fifths is produced by $(2/3)^7\ l_0 = 0.0077073\ldots$, which is higher since the length is shorter, as one can see in the spiral presented in the diagram of Figure 1. So, one must find a smaller fifth, so that the composition of twelve such fifths fits the composition of seven octaves. Supposing the ratio r underlies each of these fifths, this procedure would translate mathematically to inserting geometrically twelve lengths between l_0 and $(1/2)^7 l_0$, thus resulting in the ratio $r = (1/2)^{7/12}$ underlying the fifth, since $l_0.r^{12} = (1/2)^7 l_0$. Since $(1/2)^{7/12}$ is approximately $0.66742 > 0.66666\ldots = (2:3)$, which is the pure fifth, the tempered fifth is, as expected, a little bit smaller than the Pythagorean fifth. From this, one can deduce that the tempered fourth is an octave subtracted from a fifth which is mathematically $(1:2)/(1:2)^{7/12} = (1:2)^{5/12}$ and so on.

Since any interval is an integer number of semitones, if one ensures that all semitones are equal, namely produced by equal ratios, given any equal intervals, they will be produced by equal ratios. Thus the equal temperament equivalently consists of adjusting twelve equal semitones inside of an octave. Supposing the ratio p underlies each semitone, such a procedure translates mathematically to inserting geometrically eleven lengths between l_0 and $(1/2)\ l_0$, thus resulting in the ratio $p = (1/2)^{1/12}$ underlying the half tone, since $l_0.p^{12} = (1/2)l_0$. The equal temperament demanded symmetry, and its mathematical systematization implies the use of incommensurable magnitudes as the mathematical foundation, directly associated with musical intervals. Symmetry here means scales in which all the fifths underlie the same mathematical ratio, the same happening for any other interval. Such a use conflicts with the characteristic philosophical doctrine of the Pythagorean school, which believed in the importance of numbers –- in this case natural numbers – as a guide to the interpretation of the world.

2.1 Equal division of the tone

The equal division of the tone played an important part in the historical process that led to the emergence of equal temperament. Mathematically, the equal division of the tone 8:9 provides incommensurable ratios underlying musical intervals. The equal division of the tone (8:9) mathematically means to find x so that 8:x = x:9; anachronistically speaking, that result in irrational numbers, inconceivable in the Pythagorean musical system.

Attempts to divide the tone were already made in Antiquity, for instance by Aristoxenus (fourth century B.C.). In contrast with the Pythagoreans, who defended the position that musical intervals could properly be measured and expressed only as mathematical ratios, Aristoxenus rejected this position, asserting instead that the ear was the sole criterion of musical phenomena [592,24]. In preferring geometry to arithmetic in solving problems involving relations between musical pitches, Aristoxenus also sustained against the Pythagoreans, the possibility of dividing the tone into two equal parts, conceiving musical intervals – and indirectly ratios – as one-dimensional and continuous magnitudes, making their division possible in this way. This idea provoked a large number of reactions expressed for instance in the *Sectio Canonis* [125,5], which was in Antiquity attributed to Euclid and much later in the *De institutione musica* [88,8] of Boethius in the early Middle Ages, which gave birth to a strong Pythagorean tradition in theoretical music throughout the Middle Ages. Following the Platonic-Pythagorean tradition, a great number of medieval musical theorists sustained the impossibility of the equal division of the tone, which would mathematically lead to incommensurable ratios underlying musical intervals. Gradually, the need to carry out the temperament gave birth to different attempts to divide the tone.

Goldman suggests that Nicholas Cusanus (1401-1464) was the first to assert in *Idiota de Mente* that the musical half-tone is derived by *geometric division* of the whole-tone, and hence would be defined by an irrational number [308,4] (Goldman, 1989, 308). As a consequence, Cusanus would be the first to formulate a concept that set the foundation for the equal temperament proposed in the work of the High Renaissance music theorists Faber Stapulensis (1455-1537) and Franchino Gafurius (1451-1524), published half a century later [308,4]. Nevertheless, in the Byzantine tradition Michael Psellus (1018-1078) suggested in his *Liber de quatuor mathematicis scientijs, arithmetica, musica, geometria, [et] astronomia* [22] a geometrical division of the tone, whose underlying conception implies an understanding of ratio as a continuous magnitude. Also concerning the division of the tone before Cusanus, Marchettus of Padua (1274 ? –?) proposed, in his *Lucidarium in Arte Musice Planae* written in 1317/1318, the division of the tone into five equal parts [193,3], an innovation of extraordinary interest which made Marchettus the first in the Latin tradition to propose such a division, but without any mathematical approach. At the end of the fifteenth century and the beginning of the sixteenth century, Erasmus Horicius, one of the German humanists gifted in musical matters, wrote his *Musica* [fo. 66^{v},11], where he suggested a geometric division of the whole tone [160,16]. Erasmus stated that any part of any superparticular ratio can be obtained, in particular the half of 8:9, which corresponds to equally divide the whole tone [159,16]. Theoretically based on many geometrical propositions and, unusually, modeled on Euclidean style, his *Musica* dealt with ratio as a continuous quantity, announcing perhaps what would emerge as a truly geometric tradition in the treatment of ratios in theoretical music contexts during the sixteenth century. Such a change from an arithmetical to a geometrical basis in the theory of music represents a meaningful structural transformation in the basis of theoretical music and in the concept of ratio seen as a continuum.

2.2 The mathematical basis of Renaissance theoretical music: from arithmetic to geometry

The period from the end of the fifteenth century to the end of the sixteenth century witnessed more intense structural changes in the conceptions underlying ratios and proportions in the contexts of theoretical music. With the need of equal temperament which brings together the need of the division of the whole tone and consequently structural changes in the conceptions of ratios, treatments with such concepts in theoretical music ceased to be a subject exclusively of arithmetic and became a subject of geometry.

In this context, Erasmus Horicius contributed immensely to the introduction of geometry as an instrument for solving structural problems in theoretical music. Notwithstanding the announcements of the need for geometry in theoretical music by previous authors, Erasmus could be considered the first in the Renaissance to apply Euclidean geometry extensively in his *Musica* [14] for the resolution of structural problems in theoretical music. Relying mainly on books V and VI of Euclid, Horicius used geometry in different ways to solve musical problems, applying it to intervals, in contradiction to the Boethian arithmetical tradition. In his *Musica*, he used the *denominatio* terminology taken from Campanus's Latin translation of *the Elements*, a procedure which contributed to the emergence of an arithmetical theory of ratio in the context of theoretical music. Making use of geometrical resources hitherto unusual in musical contexts, Erasmus showed that the intervals of the fifth (3:2) and the whole tone could be divided by a proportional mean, namely by finding a magnitude *b* between *a* and *c* so that *a:b* is proportional to *b:c* considering the whole tone mathematically expressed by *a:b*, although such resources involved potentially irrational numbers. Procedures like those in musical contexts intensified the conflicts associated with the Pythagorean tradition concerning theoretical music, according to which only whole numbers and ratios of whole numbers could serve as the basis for theoretical music, whether through a stiff distinction between consonance and dissonance defined by the first four numbers or through the search for a perfect system of intonation based on commensurable ratios.

Erasmus represents an intensification in the conceptual change undergone by theoretical music at this time, and his contribution is relevant to the research on mathematics and music at the end of the fifteenth century and beginning of the sixteenth century at the University of Paris, inasmuch as one can find the use of geometry in the solution of musical problems, for instance in the geometric division of superparticular intervals (produced by an epimoric ratio, i.e. a ratio of the type m+1:m) presented in Faber Stapulensis's *Elementalia musica*, first published in 1494. This work had influence in the Spanish tradition of theoretical music in the sixteenth century, with authors like Pedro Ciruelo (1470-1548) and Juan Bermudo (1510-1565), who also presented respectively in the works *Cursus quatuor mathematicarum artium liberalium*, published in Alcalá de Henares in 1516 [11] and *Declaración de Instrumentos,* published in 1555 in Osuna [25] the same division of the tone with the geometrical mean presented by Faber Stapulensis. In the Iberian Peninsula, the tendency to use geometry occurred also in Salinas's *De Musica* published in Salamanca in 1577, that contains a geometrical systematization for the equal Temperament, which makes extensive use of Euclid's *Elements*.

Such a tendency spread also to the German and Italian production in theoretical music. The German mathematician Heinrich Schreiber (1492-1525), for instance, published in the appendix *Arithmetica applicirt oder gezogen auff die edel kunst Musica* of his "Ayn new kunstlich Buech…" of 1521 [9] a geometric division of the tone into two equal parts making, use of the Euclidean method for finding the geometric mean. He also operated with ratios with a very arithmetical structure, for instance, anachronistically compounding them as one multiplies fractions.

In the Italian tradition the tendency to use geometry was also strong. A representative example of such a tendency is Gioseffo Zarlino, a leading Italian theorist and composer in the sixteenth century. One of the most important works in the history of music theory, Zarlino's *Le istitutioni harmoniche* (1558), represents an important attempt to unite speculative theory with the practice of composition on the grounds that "music considered in its ultimate perfection contains these two parts so closely joined that one cannot be separated from the other [646,17]. The tendencies for reconciling theory and practice also manifested themselves in this period in the context of structural problems underlying theoretical music. Such a reconciliation seemed to be incompatible with a Pythagorean perspective on theoretical music, in which there was no place for geometry, an essential tool for modeling a new language claimed by practical music. This tendency also shows the consideration of number as a continuum in the treatment of theoretical music problems.

In this context, it is worthwhile to mention Zarlino's *Sopplimenti musicali* (1588), in which the Italian theorist demonstrated much greater penetration into the ancient authors, particularly Aristoxenus and Ptolemy, than in *Le istitutioni* *harmoniche* [648,17]. In spite of the still existing authority of Pythagoreanism in the context of theoretical music in the sixteenth century, Zarlino's *Sopplimenti musicali* already gave evidence of the tension between speculative theory and practice in the contexts of structural problems in theoretical music, inasmuch as it presented geometrical solutions for the equal temperament but was also based on Pythagorean foundations.

Zarlino proposed a theoretical accomplishment for the equal temperament displayed on the lute, which is presented in chapter 30 of book 4 of the third volume of Zarlino's *Sopplimenti musicali* [208,25]. Entitled *Come si possa dirittamente diuidere la Diapason in Dodici parti ò Semituoni equali & proportionali*, this chapter presented the first theoretical possibility for the equal temperament as the *temperament of the lute, made by tones and semitones, equally made in the division of the diapason (octave), in twelve proportional parts, distributed between the keys of the lute.*

It is worthwhile to mention the assertion made in the last page of the *Sopplimenti musicali*. At the end of chapter 32 of book 8 of the third volume, Zarlino wrote *"... che la Musica più tosto sia sottoposta alla Geometria, che alla Arithmetica ..."* [330,25], which means that music should be subordinate to geometry rather than to arithmetic. Zarlino published the *Sopplimenti musicali* just before his death in February of 1590. This passage in his last work seems to be the first time that Zarlino assumed explicitly that geometry was not just a theoretical tool together with arithmetic for dealing with problems in theoretical music, but rather constituted a better tool for this task then arithmetic.

CONCLUSIONS

The sixteenth century saw, in distinction to the Pythagorean tradition, the introduction of geometry as a tool not only to solve the problem of division of the tone but also to solve theoretical problems related to the systematization of the temperament as well as the emergence of the idea of irrationality in theoretical music contexts, symbolizing a substantial change in the foundations of theoretical music.

The intensification of the skepticism concerning Pythagorean arithmetical dogmatism in music at the end of the sixteenth century and the beginning of the seventeenth resulted in interest concerning the physical determinants of musical concepts. Vincenzo Galilei played an important part in this process inasmuch as he raised the paradox, under a Pythagorean perspective, that many ratios could underlie a given musical interval. Also, in 1638, based on the fact that sound had the nature of vibration, Galileo Galilei established that the direct and immediate explanation for musical intervals was neither the length or thickness of the string, nor the tension to which it was subject, but rather the ratio of the number of vibrations and impacts of waves of air that directly hit the ear (Cohen, 1984, 90). This explanation would serve as the basis for the coincidence theory of consonance that makes the old question concerning the cause of the human sensation of beauty and pleasure in hearing the consonant intervals undergo a meaningful change in which the explanation for our sense experience turned from a speculative-mathematical ratio to an empirical-physical phenomenon.

Such facts are representative, in a wider sense, of a greater change undergone by the foundations of theoretical music throughout this period, which gradually ceased to be based on an arithmetical dogmatism and assumed instead experimental principles as its basis, a change which conferred upon music the character of experimental science.

In the seventeenth century, the concept of harmony in the sense of harmony of the world and harmony of the celestial bodies, became suspect for rationalists and exact scientists. In spite of the new conceptions of theoretical music, Pythagorean ideas were still present in such contexts even in the seventeenth century, when such ideas seem in other respects to have been less present and less prominent. A representative example of such a presence can be found in book V of Kepler's *Harmonices mundi* (1619), where the Platonic-Pythagorean cosmos received a magnificent restatement, before being withdrawn. Postulating the old Pythagorean doctrine of the music of the spheres for the first time polyphonically rather than as a Greek scale, and simultaneously worrying about empirical tests for his hypotheses, Kepler finds harmonious ratios expressible in musical terms in the relationship between the speeds of revolution of the planets and pitches.

The ancient and medieval conceptions of the music of the spheres were much broader and more speculative, including not only the harmony in the course of the planets but also in the course of time, in the combination of elements, etc. Such an approach in this Pythagorean doctrine culminated in the seventeenth century with cosmic constructions such as the *monochord mundi* of Robert Fludd. This procedure possibly represents a last evidence of the Pythagorean speculative tradition in music, rescuing the old doctrine of the music of the spheres; on the other hand, it also equips the old doctrine with a mathematical-empirical conception under which, in a wider sense, western music had been approached since the late Middle Ages.

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