Homotopy Analysis Approach of Boussinesq Equation for Infiltration Phenomenon in Unsaturated Porous Medium

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Boussinesq’s equation is one-dimensional nonlinear partial differential equation which represents the infiltration phenomenon. This equation is frequently used to study the infiltration phenomenon in unsaturated porous medium. Infiltration is the process in which the groundwater of the water reservoir has entered in the unsaturated soil through vertical permeable wall. An approximate analytical solution of nonlinear partial differential equation is presented by homotopy analysis method. The convergence of homotopy analysis solution is discussed by choosing proper value of convergence control parameter. The solution represents the height of free surface of infiltrated water.

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1. Introduction

The fluid flow in porous media has great importance in various fields of engineering and science. In particular, the groundwater flow is very important part of fluid mechanics, hydrology, water resources engineering, irrigation engineering, etc. [15, 19, 20, 22, 23]. In this work, we examined the fluid flow problem in groundwater infiltration. Infiltration is the process in which water on the ground surface enters into unsaturated soils and pass into rocks through cracks and interstices. If the storage for the additional water had been available then the infiltration process can continue for a long time. The availability of additional water into the soil is dependent on the porosity of the soil. Once water has infiltrated into the soil it may stay in soil until it gradually evaporated, absorbed by plant roots and later transpired. The rate of infiltration process is dependent on different factors like as texture and structure of soil, storage capacity of soil, the depth of water reservoirs, the amount of plant over the region, etc.

Many researchers have been discussed various problems of groundwater infiltration like as Troch et al. [13] have derived an expression for mean water table height on the basis of hydraulic groundwater theory by means of Boussinesq equation, Govindaraju and Koelliker [3] have developed the expression for the flow rate from the stream to the aquifer, Hogarth et al. [4] have discussed an analytical approach for Boussinesq equation with constant and time dependent boundary conditions, Hogarth et al. [5] have obtained the approximate analytical solution of Boussinesq equation which is accurate solution by comparison with the numerical solution when the boundary conditions is a power to time, Wojnar [14] has discussed the Boussinesq equation for flow in the aquifer with time dependent porosity, Moutsopoulos [7] has discussed Boussinesq equation with nonlinear robin boundary condition, Basha [1] has discussed the traveling wave solution of the groundwater flow in horizontal aquifers.

The aim of current work is to obtain the solution of Boussinesq equation for infiltration phenomenon. The mathematical form of the infiltration phenomenon gives the nonlinear partial differential equation in the form of Boussinesq equation. This equation is solved using homotopy analysis method. The BVPh package for nonlinear equations is employed to interpret numerically and graphically solution. Liao [24] has employed the homotopy analysis method to solve nonlinear equations.
It has been successfully employed to solve many nonlinear equations. The homotopy analysis solution is strongly dependent on convergence control parameter and its proper value chosen from the valid region of \( c_0 \). The valid region of \( c_0 \) is obtained from the \( c_0 \)-curve. The line segment almost parallel to horizontal line in \( c_0 \)-curve gives us the admissible range of \( c_0 \).

2. Mathematical formulation

Let volume \( V \) of the fluid in an aquifer bounded by the surface \( \partial V \). The mass of fluid flowing in unit time through an element \( ndA \) of the surface is \( \rho u \cdot ndA \). The mass conservation law is expressed by the equation

\[
- \frac{\partial}{\partial t} \int_V PdV = \int_{\partial V} \rho u \cdot ndA
\]

(1)

where \( P \) is the porosity of the medium, \( \rho \) is the density of the fluid, \( u \) is the velocity of the fluid and \( n \) is the outward normal to the surface element. We assume that the fluid is incompressible then \( \rho \) is constant and by divergence theorem (1) becomes

\[
- \frac{\partial}{\partial t} \int_V PdV = \int_V \nabla \cdot udV.
\]

(2)

Let us assume the plane \( O_x \) be situated at the horizontal bottom of an aquifer and the \( z \)-axis be vertical up. Let \( V \) be the volume of the irregular cylinder, \( z = 0 \) is lower surface and the free surface \( z = h(x,y,t) \) is the upper surface. Then

\[
- \frac{\partial}{\partial t} \int_0^h \int_S PdzdS = \int_0^h \int_S \nabla \cdot udz dS
\]

(3)

or

\[
\int_0^h \left( \frac{\partial}{\partial t} Pdz + \int_0^h \nabla \cdot udz \right) dS = 0.
\]

(4)

The integrand under \( \int_S dS \) must be zero because the equation hold for any surface \( S \). So

\[
\frac{\partial}{\partial t} \int_0^h Pdz = -\int_0^h \nabla \cdot udz.
\]

(5)

Now we assume that the flow in unsaturated zone for which the pressure \( P_0 \) at the free surface \( z = h(x,y,t) \) is constant. Thus the vertical component of the momentum equation gives

\[
- \frac{\partial P}{\partial z} - \rho g = 0.
\]

(6)

Since the pressure \( P_0 \) at free surface \( z = h(x,y,t) \) is constant. Then we get

\[
p = P_0 + \rho g \left( h - z \right)
\]

(7)

it means that the vertical velocity component is neglected and the flow is plane.

The velocity \( u \) of the fluid obeys Darcy’s law which relates with the pressure gradient [15, 16, 20]

\[
u = -\frac{K}{\delta} \nabla p
\]

(8)

where \( K \) is the permeability of porous medium, \( \delta \) is the viscosity of water. Using (7) in (8) we get

\[
u_x = -\frac{\rho gK}{\delta} \frac{\partial h}{\partial x}, \nu_y = -\frac{\rho gK}{\delta} \frac{\partial h}{\partial y}, \nu_z = 0.
\]

(9)

So,

\[
u = -\frac{\rho gK}{\delta} \nabla h
\]

(10)

where \( \nabla h \) is the two dimensional gradient of \( h \).

Since the flow is considered in the plane then \( u \) is independent of \( z \) and the porosity \( P \) of the porous medium is constant. Thus (5) gives us

\[
\frac{\partial (Ph)}{\partial t} = -h \nabla \cdot u.
\]

(11)

Combine (10) and (11) we get

\[
p \frac{\partial h}{\partial t} = \rho \frac{gK}{\delta} h \nabla^2 h
\]

(12)

where \( K \) is the constant permeability of the porous medium.

We neglected squares of the slope angles, we get the approximation

\[
h \nabla^2 h = \nabla (h \nabla h) - (\nabla h)^2 \approx \nabla (h \nabla h).
\]

(13)

Thus (12) write as
we get the Boussinesq equation

$$p \frac{\partial h}{\partial t} = \frac{\rho g K}{\delta} \nabla (h \nabla h)$$

(14)

Using dimensionless variables

$$H = \frac{h}{l}, X = \frac{x}{l}, T = \frac{\rho g K t}{l^2 \delta}$$

(16) becomes

$$\frac{\partial H}{\partial T} = H \frac{\partial^2 H}{\partial X^2} + \left[ \frac{\partial H}{\partial X} \right]^2.$$  

(17)

This one dimensional nonlinear partial differential equation is the governing equation for infiltration phenomenon. The homotopy analysis method is applied to solve this equation with the following boundary conditions. The solution $H(X,T)$ of equation gives us the height of the free surface at a length $X$ and time $T$. According to physical concepts of the problem we used the boundary conditions are as

$$H(0,T) = h_0, \ H(1,T) = \left(h_0 - \frac{T}{2}\right) e^{-1}. \quad \text{(18)}$$

3. Homotopy Analysis Solution

Let us assume the nonlinear differential equation

$$\mathcal{N}[\phi(X,T;q)] = 0,$$

where $\mathcal{N}$ is a nonlinear operator. Let us define the nonlinear operator according to (17) is as

$$\mathcal{N}[\phi(X,T;q)] = \phi(X,T;q) \frac{\partial^2 \phi(X,T;q)}{\partial X^2} + \left[ \frac{\partial \phi(X,T;q)}{\partial X} \right]^2 - \frac{\partial \phi(X,T;q)}{\partial T} \quad \text{(19)}$$

Figure 1. The infiltration phenomenon.
where \( \varphi(X,T;q) \) is an unknown function of spatial variable \( X \) and temporal variable \( T \) for \( 0 \leq q \leq 1 \).

The zero order deformation equation is of the form [24]

\[
(1-q)L[\phi(X,T;q) - H_0(X,T)] = c_0qN[\phi(X,T;q)]
\]

(20)

where \( q \in [0,1] \) the homotopy parameter, \( L \) the linear operator; \( H_0(X,T) \) the initial approximation of the original solution \( H(X,T) \), \( c_0 \) the non-zero convergence control parameter.

Assume that the linear operator of the form

\[
L[\phi(X,T;q)] = \frac{\partial^2 \phi(X,T;q)}{\partial X^2}
\]

and the initial approximation of \( H(X,T) \) is of the form \( H_0(X,T) = \left[h_m - \frac{TX^2}{2}\right]e^{-X} \).

Thus when \( q = 0 \), (20) becomes \( \varphi(X,T;0) = H_0(X,T) \) and when \( q = 1 \), we get \( \phi(X,T;1) = H(X,T) \). Hence as \( q \) varies from 0 to 1, \( \phi(X,T;q) \) continuously varies from initial approximation \( H_0(X,T) \) to the exact solution \( H(X,T) \). The series solution is assumed in the form of

\[
\phi(X,T;q) = H_0(X,T) + \sum_{n=1}^{\infty} H_n(X,T)q^n
\]

(21)

where

\[
H_n(X,T) = \frac{1}{m!} \frac{\partial^n \phi(X,T;q)}{\partial q^n} \bigg|_{q=0}
\]

(22)

Consider the linear operator, the initial approximation and the convergence control parameter in such a way that the series of \( \phi(X,T;q) \) converges at \( q = 1 \). Thus

\[
H(X,T) = H_0(X,T) + \sum_{n=1}^{\infty} H_n(X,T).
\]

(23)

Define \( H_n(X,T) = \{H_0(X,T), H_1(X,T), ..., H_n(X,T)\} \).

Differentiating (20) \( m \) times w.r.t. \( q \) and dividing them by \( m! \) and then putting \( q = 0 \), we have the \( m^{th} \)-order deformation equation

\[
L[H_m(X,T) - \chi_m H_{m-1}(X,T)] = c_0 \mathcal{R}_m(\overline{H_{m-1}})
\]

(24)

subject to the boundary conditions

\[
H_m(0,T) = 0, H_m(1,T) = 0, \quad m \geq 1
\]

(25)

where

\[
\mathcal{R}_m(\overline{H_{m-1}}) = \sum_{i=0}^{m-1} \left[ \frac{\partial^i H_{m-1}}{\partial X^i} - \frac{\partial H_{m-1}}{\partial T}, m \geq 1 \right]
\]

(26)

and

\[
\chi_m = \begin{cases} 0, & \text{if } m \leq 1 \\ 1, & \text{if } m > 1. \end{cases}
\]

(27)

So the special solution of (24) is of the form

\[
H_m(X,T) = \chi_m H_{m-1}(X,T) + c_0 \int \mathcal{R}_m(\overline{H_{m-1}})dX.
\]

Thus the general solution (24) is of the form

\[
H_m(X,T) = \chi_m H_{m-1}(X,T) + c_1 \int \mathcal{R}_m(\overline{H_{m-1}})dX + C_1X + C_2
\]

(28)

where \( C_1 \) and \( C_2 \) are determined by boundary conditions (25). From (25) and (28), we now successively obtain

\[
H_0(X,T) = \left[h_m - \frac{TX^2}{2}\right]e^{-X}.
\]

(29)
In this way we get $H_j(X,T), H_j(X,T), \ldots$. The solution of (17) is as

$$H(X,T) = H_0(X,T) + H_1(X,T) + H_2(X,T) + \cdots \quad (31)$$

which represents the height of the free surface of infiltrated water at $X$ for a given time $T$.

4. Result and Discussion

The solution contains the convergence control parameter $c_0$ and the proper value of the convergence control parameter $c_0$ gives us the convergent homotopy series solution. The proper value of $c_0$ is chosen from the $c_0$-curve [6, 8, 9, 10, 11, 12, 17, 21, 24]. The line segment almost parallel to horizontal axis in $c_0$-curve gives us the admissible range of $c_0$. We plotted the $c_0$-curves with the help of Mathematica BVPh package for homotopy analysis method [18]. Figure 2-3 show the $c_0$-curves of $H_{xx}(0,0), H_{xx}(0.8)$ and $H_{xx}(0,1)$ for 10th and 15th order of approximation, respectively. The admissible range of $c_0$ is $-1.3 \leq c_0 \leq -0.4$ from $c_0$-curves.

Table 1 shows the values of $H_{xx}(0,0), H_{xx}(0.8)$ and $H_{xx}(0,1)$ for different values of $c_0$ and for different order of approximations using homotopy analysis solution.

Figure 2. The $c_0$-curves of $H_{xx}(0,0)$ (DotDashed line), $H_{xx}(0,0.8)$ (Thick line) and $H_{xx}(0,1)$ (Dashed line) for $h_m = 0.98$

Figure 3. The $c_0$-curves of $H_{xx}(0,0)$ (DotDashed line), $H_{xx}(0,0.8)$ (Thick line) and $H_{xx}(0,1)$ (Dashed line) for $h_m = 0.98$.
Table 1. Approximations of $H_{XX}(0,0)$, $H_{XX}(0,0.8)$ and $H_{XX}(0,1)$ for different values of $c_0$ and for different order of approximations using homotopy analysis solution.

<table>
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<tr>
<th>$c_0$</th>
<th>$10^{th}$ order</th>
<th>$15^{th}$ order</th>
<th>$10^{th}$ order</th>
<th>$15^{th}$ order</th>
<th>$10^{th}$ order</th>
<th>$15^{th}$ order</th>
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Table 2. Numerical values of the height of infiltrated groundwater $H(X,T)$.

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<td>0.79633</td>
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<tr>
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**Figure 4.** The graph of the height of infiltrated groundwater $H(X,T)$ v/s length $X$ for fixed time $T=0.1$ (uppermost graph), $0.2$, ..., $1$ (lowermost graph) for $h_m = 0.98$.

**Figure 5.** The graph of the height of infiltrated groundwater $H(X,T)$ v/s $X$ and $T$ for $h_m = 0.98$.

**Table 3:** Difference between 15th order and 10th order approximation of $H(X,T)$.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$X = 0.1$</th>
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<th>$X = 0.4$</th>
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<th>$X = 0.7$</th>
<th>$X = 0.8$</th>
<th>$X = 0.9$</th>
<th>$X = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-2.0811E-06</td>
<td>-5.6006E-06</td>
<td>-1.9778E-05</td>
<td>-5.3297E-05</td>
<td>-5.9018E-05</td>
<td>3.5795E-05</td>
<td>1.4286E-04</td>
<td>9.3097E-06</td>
<td>-1.1770E-04</td>
<td>3.8568E-08</td>
</tr>
</tbody>
</table>
The difference of 15th order approximation of solution \( H(X,T) \) and 10th order approximation of solution \( H(X,T) \) are mentioned in table 3 for different values of \( X \) and \( T \).

Conclusions

The Boussinesq equation is discussed for infiltration phenomenon in unsaturated soil. The homotopy analysis solution of the governing equation is obtained with boundary condition. The convergence of homotopy analysis solution is discussed by \( c \)-curve. The solution represents the height of free surface which is discussed graphically and numerically.

References


