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Maximal Left Ideals In Local Goldie (-1, 1) Rings

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Abstract In this paper we show how to reduce the study of nondegenerate local Goldie (-1, 1) rings to the strongly prime case, via the notions of uniform ideals and essential subdirect product. Also we construct the maximal left quotient ring of (-1, 1) ring that is a left quotient ring of itself. We follow Utumi where a maximal left quotient ring is constructed as a direct limit of partially defined homomorphism from left ideal of R to R.

Keywords: (-1, 1) rings, Quotient rings, local Goldie ring, nondegenerate.

1. INTRODUCTION

Goldie Theorem is certainly one of the fundamental results of the theory of associative rings. Today this Theorem is usually formulated as follows: A ring R is a classical left order in a semisimple (equivalently, semiprime artinian) ring Q if and only if R is semiprime, left nonsingular, and does not contain infinite direct sums of left ideals. Moreover, R is prime if and only if Q is simple. Whereas the theory of rings of quotients has its origins between 1930 and 1940 in the works of Ore and Osano on the construction of the total ring of fractions. In that decade Ore proved that a necessary and sufficient condition for a ring R to have a (left) classical ring of quotients is that for all regular element $a \in R$ and $b \in R$ there exist a regular $c \in R$ and $d \in R$ such that $cb = da$ (left Ore condition). This Ore conditions were also used by author in [9] to study the properties of ascending chain conditions in (-1, 1) rings.

Fountain and Gould (1990), based on ideas from semigroup theory, introduced a notion of order in a ring, which need not have an identity, and in the following years gave a Goldie – like characterization of two – sided orders in semiprime rings with descending chain condition on principal one – sided ideals (equivalently, coinciding with their socles). Anh and Marki (1991) extended this

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result to one – sided orders, and in (1994) the same authors developed a general theory of Fountain – Gould left quotient rings (we point out that the maximal left quotient ring plays a fundamental role in this work).

It is natural to ask whether similar notions (and results) can be obtained for (-1, 1) rings. In this work we show how to reduce the study of nondegenerate local Goldie (-1, 1) rings to the strongly prime case, via the notions of uniform ideal and essential subdirect product. We shall also introduce as a tool, the notion of general left quotient rings and related properties of a ring to any of its general rings of quotients. We construct the maximal left quotient ring of (-1, 1) ring that is a left quotient ring of itself and prove that this is a (-1, 1) ring when $A(R)$ is semiprime or 2-torsion free. We finish giving explicitly the maximal left quotient ring of particular (-1, 1) rings.

2. PRELIMINARIES

The following three basic subsets can be considered in a nonassociative ring R: the nucleus $N(R)$, the commutative center $C(R)$ and the center $Z(R)$ defined by $N(R) = \{n \in R \,/\, (n, R, R) = (R, n, R) = (R, R, n) = 0\}$, also known as associative center $C(R) = \{c \in R \,/\, [c, R] = 0\}$ and $Z(R) = N(R) \cap C(R)$ where $[x, y] = xy - yx$ denotes the commutators of two elements $x, y \in R$ and $(x, y, z) = (xy)z - x(yz)$ is the associator of the three elements x, y, z of R.

The defining axioms of (-1, 1) ring R are $(x, y, y) = 0$ (2.1)

and $S(x, y, z) = 0$ for every $x, y, z \in R$ (2.2)

where after linearization of (2.1) we obtain $(x, y, z) + (x, z, y) = 0$ and $S(x, y, z)$ is nothing but $(x, y, z) + (y, z, x) + (z, x, y)$. In fact there are many results where we can see a (-1, 1) ring becoming an alternative ring. The standard reference for alternative rings is [12].

From now on, for a ring R, R^1 will denote its unitization, that is, R if the ring is unital, or $R \times \mathbb{Z}$ with product $(x, m)(y, n) := (xy + nx + my, mn)$ if R has no unity.

The nucleus and the associator ideal of a (-1, 1) ring will be very important notions in this theory. Given a ring R, every ideal contained in the nucleus of R will be called a nuclear ideal. The largest nuclear ideal of R will be the associative nucleus denoted by $U(R)$ where $U = U(R) = \{x \in R \,/\, xR^1 \subseteq N(R)\} = \{x \in R \,/\, R^1x \subseteq N(R)\}$. By $A(R)$ we will mean the associator ideal, i.e., the ideal of R generated by the set (R, R, R) of all associators.

If X is a nonempty set of a (-1, 1) ring R, then the left annihilator of X is defined to be the set $lan(X) = \{a \in R/\ ax = 0 \text{ for all } x \in X\}$, written $lan_R(X)$

when it is necessary to emphasize the dependence on R. Similarly the right annihilator of R $ran(X) = ran_R(X)$, is defined by $ran(X) = \{a \in R / xa = 0$ for all $x \in X\}$.

We also write $ann(X) = ann_R(X) = lan(X) \cap ran(X)$ to denote the annihilator of X. It is easily seen that if X is the left (right) ideal of R, then $ran(X)$ ($lan(X)$) is a right (left) ideal of R, and if X is an ideal, then $lan(X)$, $ran(X)$ and $ann(X)$ are ideals of R.

For every subset X of R we have the third annihilator property :

$$lan(ran(lan(X))) = lan(X) \text{ and } ran(lan(ran(X))) = ran(X).$$

A ring without nonzero trivial ideals (i.e., ideals with zero multiplication) is called semiprime. By [8] every semiprime alternative ring does not contain nonzero trivial left (right) ideals and hence so is in the case of (-1, 1) rings. An element a of a (-1, 1) ring R is called an absolute zero divisor if $aRa = (0)$. The ring R is nondegenerate (or strongly semiprime) if R does not contain nonzero absolute zero divisors.

Proposition 2.1: Let I be an ideal of a semiprime (-1, 1) ring R and denote $\pi : R \rightarrow \overline{R}$ the canionical projection of R onto $\overline{R} = R / ann(I)$. Then

(i) $lan(I) = ran(I) = ann(I)$ is a two sided ideal of R.

(ii) $I \cap ann(I) = 0$.

(iii) $\overline{R}$ is a semiprime (-1, 1) ring.

(iv) I is an essential ideal of R if and only if $ann(I) = 0$.

(v) $\overline{I} = \pi(I)$ is an essential ideal of $\overline{R}$.

If R is nondegenerate, then

(vi) $ann(I) = \{x \in R / xIx = 0\}$.

(vii) $\overline{R}$ is a nondegenerate (-1, 1) ring.

Proof: (i) Let $x \in Lan(I)$. We will see that $x \in ran(I)$.
For every $r \in R$ and $y \in I$,

(a) $r(yx) = -(r, y, x) + (ry)x$
$= (y, x, r) + (x, r, y) + (ry)x$. ($S(x, y, z) = 0$ i.e., from (2.2))
$= -(y, r, x) - (x, y, r) + (ry)x$ from (2.1)
$= (ry)x$.

(b) $(rx)y = (r, x, y) + r(xy)$
$= -(x, y, r) - (y, r, x)$ from (2.2)
$= -(x, y, r) = 0$ from (2.1)

From (a), Ix is a left ideal of R and from (b), $(Ix)^2 = 0$. Since R is semiprime, $Ix = 0$.

Similarly we prove $ran(I) \subseteq lan(I)$

(ii) We have $(I \cap ann(I))^2 \subseteq ann(I) = 0$, and by semiprimeness of R we obtain $I \cap ann(I) = 0$.

(iii) Let $\bar{J} = \pi(J)$ be an ideal of $\bar{R}$, with J an ideal of R, such that $(\bar{J})^2 = \bar{0}$. Since $J^2 \subseteq ann(I)$, $(I \cap J)^3 \subseteq J^2I + IJ^2 \subseteq ann(I) = 0$ and semiprimeness of R implies $I \cap J = 0$. In consequence $IJ = 0 = JI$, which implies $\bar{J} = \bar{0}$.

(iv) If I is an essential ideal of R, then from (ii) $I \cap ann(I) = 0$ implies $ann(I) = 0$. Conversely, suppose $ann(I) = 0$. If J is an ideal of R satisfying $J \cap I = 0$ then, since $IJ \subseteq I \cap J = 0$. By (i), $J \subseteq ann(I) = 0$ and hence I must be essential.

(v) By (iii), $\bar{R}$ is semiprime, so by (i), $lan_{\bar{R}}(\bar{I}) = ann_{\bar{R}}(\bar{I})$. Hence and by (iv) it is enough to prove that $lan_{\bar{R}}(\bar{I}) = \bar{0}$. Let $\bar{r}$ be in $lan_{\bar{R}}(\bar{I})$. Then $rI \subseteq I \cap ann(I)$, which is zero by (iii). And by (i), $\bar{r} = \bar{0}$.

(vi) By [4] the Jordan annihilator of I (see note after Lemma 1.3), $ann^J(I)$, which coincides with the set $\{x \in R / xIx = 0\}$, is an ideal of R such that $ann^J(I) \cap I = 0$. This implies $(ann^J(I) \cap I)^2 = 0$ since R is semiprime, $ann^J(I)\ I = 0$. Hence $ann^J(I) \subseteq lan(I) =$ (by (i)) $ann(I)$. The inclusion $ann(I) \subseteq ann^J(I)$ is obvious.

(vii) Assume that there exist $\bar{a} \in \bar{R}$ such that $\bar{a}\,\bar{b}\,\bar{a} = \bar{0}$ for every $b \in R$. In particular, for every $b \in I$ we have $aba \in I \cap ann(I)$, which is zero by (iii), and therefore $a \in ann(I)$, by (v). ♦

Every (-1, 1) ring R with $Z_l(R) = 0$ and such that every element has finite left Goldie dimension will be dimension will be called a **left local goldie (-1, 1)ring.** If additionally R has finite left (global) Goldie dimensions then R will be called left goldie (-1, 1)ring right and two-sided corresponding notions are defined dually.

Following [12], given a ring R we call every ideal contained in the associative center $N(R)$ a nuclear ideals, and the largest nuclear ideal the associative nucleus of the ring R. We denote the letter by $U = U(R)$. This ideal of R can be characterized as follows. (See [12] [proposition 8.9]).

$U = \{x \in R / xR^1 \subseteq N(R)\} = \{x \in R / R^1x \subseteq N(R)\}_\vee$

For a left ideal L of a (-1, 1) ring R, denote by L the largest ideal of R contained in L. ♦

Lemma 2.1: Let L be a nonzero left ideal of a semiprime (-1, 1) ring R. Then:

(i) N(L) = L ∩ N(R)
(ii) Z(L) = L ∩ Z(R)
(iii) U(L) = L ∩ U(R).

Proof : (i) It is sufficient to prove the assertion in the case when the ring R is prime. Let $i, j \in L$; $a \in R$, $b \in R^*$.Then by the linearized Moufang identity $b(i, j, a) = -j(i, b, a) + (i, j, ab) + (i, b, aj) \in L$ whence $(L, L, A) \subseteq (L : A^*) = \check{L}$. Now if $L = (0)$, then $(L, L, A) = (0)$, which by Lemma 5 in [12] implies $L \subseteq N(R)$. It is easy to see that in this case the assertion of the Theorem is true.

But if $L \neq (0)$ then Lemma 8.10 in [12] $T(L) \neq 0$. For any element $n \in N(L)$ we have $(n, L, L) = (0)$, whence by Lemma 9.1 in [12] $(n, R, R) \subseteq Ann_r T(L)$. Since the algebra R is prime, we obtain $Ann_r T(L) = (0)$, and consequently $n \in N(R)$. Thus the inclusion $N(L) = L \cap N(R)$ is proved. The reverse inclusion is obvious.

(ii) Let $z \in Z(L)$. Then by (i) $z \in N(R)$. Let $i \in L$, $a, b \in R$. Then by the semi Jacobi identity which is true in any arbitrary ring we see that $i[z, a] = [z, ia] - [z, i]a = 0$.

Furthermore, if we assume that our ring satisfies $(w, [z, a], y) = ([z, a], y, w) = (y, w, [z, a])$ then $[z, a] \in N(R)$ and therefore $(bi)[z, a] = b(i[z, a]) = 0$. We see that $[z, a] \in Ann_r(L)$. In views of the fact that the element $[z, a]$ belongs to the ideal L and its right annihilators it also belongs to their intersection, which is a trivial ideal in R. Since R is semiprime $[z, a] = 0$, which means that $z \in Z(R)$. We have proved that $Z(L) \subseteq L \cap Z(R)$. This proves (ii) since the reverse inclusion is obvious.

(iii) Consider $u \in U(L)$, $x, y \in L$ and $r \in R$. By (i), $u \in N(R)$ and since $U(L)$ is an ideal of L, $xu \in U(L) \subseteq N(L) = L \cap N(R)$ by (i). By applying the teichmullar identity, $[R, N(L)] \subseteq N(L)$ and the above said fact we see that $(ru, x, y) = (r, xu, y) = 0$. This implies $R^1 u \subseteq N(L) = L \cap N(R)$ by (i) and by Proposition 2.1, $u \in U(R)$. The reverse inclusion is obvious. ♦

Lemma 2.2: For every ideal I of a semiprime (-1, 1) ring R we have; $Z_l(I) = I \cap Z_l(R)$.

Proof : Assume $I \neq 0$. Take $0 \neq x \in Z_l(I)$. By condition (iii) in Lemma 2.1 $x \in U(R)$. Now we see that $lan_R(x)$ is an essential left ideal of R. Let L be a nonzero left ideal of R. If $Lx = 0$ then $L \subseteq lan_R(x)$. If $Lx \neq 0$ then IxL would be a nonzero left ideal of I (otherwise $IxL = 0$ would imply $L_x L_x L_x \subseteq IxLx = 0$, which is not possible by the semiprime of R) and $0 \neq IxL \cap lan_I(x) \subseteq L \cap lan_R(x)$.

Conversely, let $0 \neq x \in I \cap Z_l(R)$. By Lemma 2.1 (iii), $x \in U(I)$. Now we prove that $lan_I(x)$ is an essential left ideal of I. Let L be a nonzero left ideal of I. If $Lx = 0$, then $L \subseteq lan_I(x)$. If $Lx \neq 0$ then RxL is a nonzero left ideal of R ($RxL = 0$ would imply $LxLx \subseteq RxLx = 0$ and so Lx would be a nonzero trivial left ideal in a semiprime ring). Since $lan_R(x)$ is an essential left ideal of R, $0 \neq RxL \cap lan_R(x)$ and $L \cap lan_I(x)$ is nonzero. ♦

Corolary 2.1: Let R be a semiprime (-1, 1) ring and I an essential ideal of R. Then $Z_l(R) = 0$ if and only if $Z_l(I) = 0$.

Lemma 2.3: Let R be semiprime (-1, 1) ring. Then every 1-uniform element generates a uniform ideal.

Proof : Let I be the ideal generated in R by a 1-uniform element u and let J and K be nonzero ideals of R contained in I. We note that $Ju \neq 0$ otherwise $u \in ran(J) = ann(J)$ (by Preposition 2.1(i)),which implies $I \subseteq ann(J)$,and since $J \subseteq I$,we would have $J = J \cap ann(J) = 0$ (by Proposition 2.1 (ii)), a contradiction. Analogously $Ku \neq 0$. Denote the L and L' the nonzero left ideals of R generated by Ju and Ku respectively. Then $0 \neq L \cap L' \subseteq J \cap K$ by the 1-uniformity of u.

Since given a (-1, 1) ring R, the lattice *£(R)* of all ideals of R is an algebraic relative to the $*$ – product $J * K:=(JK)$, where (X) denotes the ideal of R generated by X, and XY the linear span of all products xy with $x \in X$ and $y \in Y$, we can apply the result of [7] to obtain the following result. We note that $J * K = 0$ if and only if $JK = 0$. Hence R *is semiprime* (prime) if and only if the algebraic lattice (£(R), $*$) is semiprime (prime). Note that since (-1, 1) ring is alternative in many cases [10], $J * K$ is merely JK. ♦

Lemma 2.4: Let R be a semiprime (-1, 1) ring. Then

(i) A nonzero ideal I of R is uniform if and only if the annihilator ideal $ann(I)$ is maximal among all annihilator ideals $ann(J)$ with J being a nonzero ideal of R, equivalently, $R/ann(I)$ is a strongly prime ring.

(ii) For each uniform ideal I of R there exists a unique maximal uniform ideal U of R containing I; actually $U = ann(ann(I))$.

(iii) The sum of all maximal uniform ideal of R is direct.

Proof: Suppose that R is nondegenerate and that $R/ann(I)$ is a prime (-1, 1) ring. By Proposition 2.1(iv), the ring $R/ann(I)$ is nondegenerate and so it is strongly prime.

The rest of the statement follows as a particular case of [7 Proposition 3.1], using that £(R) is a modular lattice. ♦

A subdirect product of (-1, 1) ring $R \leq \prod R_\alpha$ will be called an essential subdirect product if R contains an essential ideal of the full product $\prod R_\alpha$. If R is actually contained in the direct sum of the R_α, then R will be called an essential subdirect sum. An ideal I of a nondegenerate (-1, 1) ring R is called a closed ideal if $I = ann(ann(I))$. By the third annihilator property an ideal is closed if and only if it is the annihilator of an ideal. Note that by the above Lemma 2.4 (ii), maximal uniform ideals are closed.

Theorem 2.1: For a (-1, 1) ring R the following conditions are equivalent :

(i) R is an essential subdirect product of strongly prime (-1, 1) rings R_α.

(ii) R is nondegenerate and every nonzero ideal of R contains a uniform ideal.

(iii) R is nondegenerate and every nonzero closed ideal of R contains a uniform ideal.

Actually we can take $R_\alpha = ann(M_\alpha)$, where $\{M_\alpha\}$ is the family of all maximal uniform ideal of R.

Proof: (i) $\Rightarrow$ (ii). In general any subdirect product R of a family $\{R_\alpha\}$ of nondegenerate (-1, 1) rings is nondegenerate. Let $M \subseteq R$ be an essential ideal of the full direct product $\prod R_\alpha$ and set $M_\alpha := M \cap R_\alpha$, where we are regarding R_α as an ideal of $\prod R_\alpha$. Then M_α is a nonzero ideal of R_α contained in R since M is an essential ideal $\prod R_\alpha$. In fact M_α is a uniform ideal of R since M_α is uniform in R_α because R_α is strongly prime and any ideal of R contained in M_α is an ideal of R_α. Now if I is a nonzero ideal of R then $\pi_\alpha(I)$ is a nonzero ideal of R_α for some index α. Hence by strongly primeness of R_α we have $0 \neq \pi_\alpha(I) * M_\alpha \subseteq I \cap M_\alpha$. Therefore I contains the nonzero ideal $I \cap M_\alpha$ which is uniform since it is contained in M_α.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) Let $\sum M_\alpha$ be the sum of all maximal uniform ideal of R, which is direct by Lemma 2.4 (iii). Since $ann(\sum M_\alpha)$ is a closed ideal, it must be zero: otherwise $ann(\sum M_\alpha)$ would contain a uniform ideal, and therefore a maximal uiform ideal because it is closed, which leads to contradiction. Hence, by a standard argument, $\bigcap ann(\sum M_\alpha) = ann(\sum M_\alpha) = 0$ implies that R is a subdirect product of the (-1, 1) rings $R_\alpha := R / ann(M_\alpha)$ each of which is a strongly prime (-1, 1) ring by Lemma 2.4 (i).

Finally, the homomorphic image of $\oplus M_\alpha$ in $\prod R_\alpha$ is an essential ideal of $\prod R_\alpha$ since M_α can be regarded as an essential ideal of R_α, by condition (v) in proposition 1. ♦

Lemma 2.5: Let R be a semiprime (-1, 1) ring and I an ideal of R. Denote by $\overline{R}$ the quotient ring $R / ann(I)$. We have:

(i) Any direct sum of nonzero left ideals of $\overline{R}$ gives rise to the direct sum of nonzero left ideals of R with the same number of summands. Hence if R has finite left Goldie dimension then $\overline{R}$ has also finite left Goldie dimension.

(ii) If $a \in R$ has finite left Goldie dimension in R, then $\overline{a} := a + ann(I)$ has finite left Goldie dimension in $\overline{R}$.

(iii) If $Z_l(R) = 0$ then $Z_l(\overline{R}) = \overline{0}$. Moreover, if R is nondegenerate and I is a uniform ideal, then $\overline{R}$ is a strongly prime left nonsingular (-1, 1) ring.

Proof: Let $\sum \overline{L}_\alpha$ be a direct sum of nonzero left ideals of $\overline{R}$. Denote by $\pi : R \to \overline{R}$ the canonical projection of R onto $\overline{R}$. By Proposition 2.1(v), $\pi(I)$ is an essential left ideal of $\overline{R}$. Hence $L_\alpha := \pi^{-1}(\overline{L}_\alpha) \cap I$ is a nonzero left ideal of R, for each index α. Now, we show that $\sum L_\alpha$ is a direct sum. Indeed, if $x \in L_\beta \cap (\sum_{\alpha\neq\beta} L_\alpha)$, then $\pi(x) \in \overline{L}_\beta \cap (\sum_{\alpha\neq\beta} \overline{L}_\alpha) = \overline{0}$. Therefore $x \in I \cap ann(I) = 0$ (by Proposition 2.1(ii)).

(ii) Let $\sum \overline{L}_\alpha$ be a direct sum of nonzero left ideals of $\overline{R}$ inside the principal left ideal. By taking $L_\alpha := I(\pi^{-1}(\overline{L}_\alpha))$ we obtain a direct sum of nonzero left ideals of R contained in $\overline{R}\ (a]$.

(iii) By Lemma 2.2, $Z_l(R) = 0$ implies $Z_l(I) = 0$. But I can be regarded as an ideal of $\overline{R}$ via the isomorphism $x \mapsto x + ann(I)$, for $x \in I$ and by Proposition 2.1 (v) we have that $\overline{I}$ is an essential ideal of $\overline{R}$. By proposition 2.1(iii) and Corollary to Lemma 2.2, $Z_l(\overline{R}) = \overline{0}$. If R is nondegenerate and I is uniform, then $\overline{R}$ is strongly prime by condition (i) in Lemma 2.4. ♦

Theorem 2.2 : Let R be nondegenerate left local Goldie (-1, 1) ring. Then R is an essential subdirect sum of strongly prime left local Goldie (-1, 1) rings. More precisely $\oplus M_\alpha \lhd R \leq \oplus R / ann(M_\alpha)$, where M_α rangs over all maximal uniform ideals of R. If R is actually left Goldie, then R is an essential subdirect sum of finitely many strongly prime left Goldie (-1, 1) rings.

Proof: Since R has finite left local Goldie dimension, any nonzero ideal of R contains an 1 - uniform element and hence a uniform ideal, by Lemma 2.3. Then by Theorem 2.1, R is an essential subdirect product of the strongly prime (-1, 1) rings $R_\alpha = R / ann(M_\alpha)$, with M_α a maximal uniform ideal of R, each of which is strongly prime left local Goldie (-1, 1) ring by Lemma 2.4 (ii) and condition (ii) and (iii) in Lemma 2.5. Let us see that $R \subseteq \oplus R_\alpha$. Otherwise, there would exist $x \in R$ such that $x \notin ann(M_\alpha)$ for an infinite number of α's. Say $x \notin ann(M_\alpha)$ for every $\alpha \in \wedge$. Denote by I_α the left ideals of R generated by $M_\alpha x$. Then $0 \neq M_\alpha x \subseteq I_\alpha \subseteq M_\alpha$ for every $\alpha \in \wedge$ and the sum $\sum_{\alpha\in\wedge} I_\alpha$ is direct. This implies that x has infinite left Goldie dimension, a contradiction.

Suppose additionally that R has finite left Goldie dimensition. Then it follows, from Lemma 2.4(iii) that R contains only a finite number of maximal uniform ideals, and hence R is an essential subdirect sum of a finite number of R_α. Moreover, each R_α has now finite left Goldie dimension by Lemma 2.5(i). ♦

3. THE CONSTRUCTION OF MAXIMAL LEFT QUOTIENT RINGS OF (-1, 1) RINGS

Definition 3.1: We shall say that a (-1, 1) ring R has a maximal left quotient ring if there exists a ring Q such that

(i) Q is a left quotient ring of R and
(ii) If W is left quotient ring of R, there exists a unique monomorphism of rings $f : W \to Q$ with $f(r) = r$ for every $r \in R$.

Clearly, this definition implies that a maximal left quotient ring of a ring R, if it exists, is unique up to isomorphisms,. We shall denote it by $Q^l_{max}(R)$.

Definition 3.2: We will say that a left ideal I of a (-1, 1) ring R is dense if for every $p, q \in R. \in R$, with $p \neq 0$, there exists an $a \in N(R)$ such that $ap \neq 0$ and $aq \in I$.

Lemma 3.1: A left ideal I of a (-1, 1) ring R is dense if and only if R is a left quotient ring of I.

Proof: Suppose that I is a dense left ideal of R. On the One hand, given $n \in N(I)$, if there exists $p, q \in R$ such that $(n, p, q) \neq 0$, then there exist $n_1, n_2 \in N(R)$ such that $n_1 p \in I, n_2 q \in I$ and $0 \neq n_2 n_1 (n, p, q) = (n, n_1 p, n_2 q) = 0$ and $0 \neq n_2 n_1 \{S(n, p, q)\} = S(n, n_1 p, n_2 q) = 0$, a contradiction. On the other hand, given $p, q \in R$, with $p \neq 0$, there exists $n \in N(R)$ such that $np \neq 0$ and $nq \in I$ and there exists $s \in N(R)$ such that $snp \neq 0$ and $sn \in I$. So, $sn \in N(R) \cap I \cap N(I)$ and $snp \neq 0$ and $snq \in I$. The reciprocal is trivial. ♦

Definition 3.3: Let R be a (-1, 1) ring. We denote F^* as the set of all left ideals A of $N(R)$ such that for every $0 \neq x \in R$ and $u \in N(R)$, there exists $\lambda \in N(R)$ such that $\lambda x \neq 0$ and $\lambda u \in A$.

Lemma 3.2: Let R be a (-1, 1) ring. Then

(i) If I is a dense left ideal of R, then $N(I) \in F^*$.

(ii) If $A \in F^*$, then $I := R^1 A$ is a dense left ideal of R.

Proof: (i) It is straight forward.

(ii) Let $x, y \in R, x \neq 0$. By hypothesis there exists $\lambda \in A$ such that $\lambda x \neq 0$. By [12 Corollary 1 of Lemma 7.1.3] $[\lambda, y] := \lambda y - y\lambda \in N(R)$, so if we apply the hypothesis again, there exists $u \in N(R)$ such that $u\lambda x \neq 0$ and $u[\lambda, y] \in A$. Therefore $u\lambda y = u[\lambda, y] + uy\lambda \in R^1 A$, which completes the proof. ♦

Notation 3.1: We call $F: = \{R^1 A \,/\, A \in F^*\}$. Now, given $I = R^1 A \in F$, we have that $A \subset N(I)$, hence $I = R^1 A \subset R^1 N(I) \subset I$. Therefore $I = R^1 N(I)$, with $N(I) \in F^*$. Moreover, the intersection of a finite family of elements of F contains an element of F as it can easily be seen that λ can be taken in A and that the intersection of a finite family of elements of F^* is an element of F^*.

Let us consider $W := \{(I, f) \,/\, I \in F$ and $f \in \mathrm{Hom}^*_{N(R)}(I, R)\}$ where $\mathrm{Hom}^*_{N(R)}(I, R)$ denotes the set of all homorphism of left $N(R)$ – modules from I to R such that for every $x \in R$ and $\lambda, u \in N(I)$, $(x\lambda)f = x(\lambda)f$ and $([\lambda, u])f \in N(R)$.

The following relation on W is an equivalence relation : $(I, f) \approx (I', f')$ if and only if there exists $I'' \in F$ such that $f|_{I''} = f'|_{I''}$ We denote by $[I, F]$ the equivalence class of (I, f) and let $Q := s/\approx$.

Abusing notation, given an element $q \in Q$, we will denote by A_q any element of F^* and by f_q any element of $\mathrm{Hom}^*_{N(R)}(R^1 A_q, R)$ such that $q = [R^1 A_q, f_q]$. The dense left ideal $R^1 A_q$ will be denote by I_q.

Let us define an $N(R)$ – algebra structure on Q: Let $q, q' \in Q$ and $\lambda, u \in N(R)$:

(i) We define the sum $q + q^1 := [R^1(A_q \cap A_{q'}), f_q + f_{q'}]$.

(ii) We define the structure of the left $N(R)$ – modules : $\lambda q := [R^1 A_{\lambda q'}, \rho_\lambda f_q]$, where $A_{\lambda q} := \{a \in N(R) / a\lambda \in A_q\} \in F^*$ and ρ denotes right multiplication.

(iii) We define the structure of the right $N(R)$ – module: $q\lambda := [I_{q,} f_q \rho_\lambda]$.

(iv) We define a product on Q: we denote by $A_{qq'} := \{ \lambda \in A_q$ such that $(\lambda) f_q \in I_{q'} \}$.

First we show that $A_{qq'} \in F^*$. It is clear that it is a left ideal of R. Now, given $0 \neq x \in R$ and $u \in N(R)$, there exists $\lambda \in N(R)$ such that $\lambda x \neq 0$ and $\lambda\mu \in A_q$, and there exists $\gamma \in N(R)$ such that $\gamma\lambda x \neq 0$ and $\gamma(\lambda\mu) f_q \in I_{q'}$, because $I_{q'}$ is a dense left ideal of R. So $\gamma\lambda \in N(R)$ and verifies that $\gamma\lambda x \neq 0$ and $\gamma\lambda\mu \in A_{qq'}$.

Now, we can define the product $qq' := [R^1 A_{qq'}, f_{qq'}]$, where $(\sum x_i a_i) f_{qq'} := \sum x_i((a_i) f_q) f_{q'}$ for every $x_i \in R^1$ and $a_i \in A_{qq'}$. Let us show that it is well defined. Suppose that $\sum x_i a_i = 0$, where $x_i \in R^1$ and $a_i \in A_{qq'}$ but $\sum x_i((a_i) f_q) f_{q'} \neq 0$. By hypothesis there exist $\mu \in A_{qq'}$ such that $\mu \sum x_i((a_i) f_q) f_{q'} \neq 0$. Then

$$\begin{aligned}
\mu\sum x_i((a_i)f_q) f_{q'} &= \sum ([\mu, x_i] + x_i\mu)((a_i)f_q) f_{q'} \\
&=^{(1)} ((\sum ([\mu, x_i]a_i)f_q) f_{q'} + \sum x_i ((\mu a_i)f_q) f_{q'} \\
&= ((\mu\sum x_i a_i)f_q) f_{q'} - ((\sum x_i\mu a_i)f_q) f_{q'} + \sum x_i ((\mu a_i)f_q) f_{q'} \\
&= -((\sum x_i[\mu a_i])f_q) f_{q'} - ((\sum x_i a_i\mu)f_q) f_{q'} + \sum x_i(([\mu, a_i])f_q) f_{q'} \\
&\qquad + \sum x_i((a_i\mu)f_q) f_{q'} \\
&=^{(2)} -((\sum x_i[\mu, a_i])f_q) f_{q'} + ((\sum x_i[\mu, a_i])f_q) f_{q'} = 0.
\end{aligned}$$

(i) is a consequence of μ and $[\mu, x_i]$ belonging to $N(R)$ and f_q and $f_{q'}$ being homorphism of left $N(R)$ – modules, and (2) uses the previous facts and $([\mu, a_i]fq \in N(R^1 Aq')$, since $([\mu, a_i]fq$ belongs to $N(R)$ and also belongs to $R^1 Aq'$ (because $[\mu, a_i] \in Aqq'$) hence $((\sum x_i[\mu, a_i])f_q) f_{q'} = \sum x_i(([\mu, a_i])f_q) f_{q'}$ by definition of $\mathrm{Hom}^*_{N(R)}(I_{q'}, R)$.

Theorem 3.1: Let Q be as above. Then

(i) R is a subring of Q. Moreover, R is a dense left $N(R)$ - Submodule of Q.

(ii) $N(R) \subset N(Q)$

(iii) If in addition R is Weakly Novikov ring then for every $q \in Q$ and $\lambda \in N(R)$, $[\lambda, q] \in N(Q)$.

(iv) The associator is a skew – symmetric function on Q.

(v) If $A(R)$ is 2 – torsion free or semiprime, Q is a (-1, 1) ring.

Proof: (i) The map $\psi : R \to Q$ given by $\psi(r) = [R^1N(R), \rho_r]$ defines a monomorphism of rings : it is clear that ψ is a monomorphism of $N(R)$ – modules, since ρ_r cannot vanish on a dense left ideal of R. Moreover, $\psi(r)\ \psi(r') = [R^1N(R), \rho_r]\ [R^1N(R), \rho_{r'}] = [R^1N(R), f_{rr'}]$, where for every $x \in R$ and $\lambda \in N(R), (x\lambda)\ (f_{rr'}) = x((\lambda)\rho_r\ \rho_{r'}) = x(\lambda rr') = (x\lambda)\rho_{rr'}$, so $\psi(r)\ \psi(r') = \psi(rr')$. Now given $q, q' \in Q$ with $q \neq 0$, by construction there exists $r \in A_q \cap A_{q'}$ such that $(r)f_q \neq 0$ (since $I_q \cap f_{q'}$ is a dense left ideal of R). Hence $[R^1N(R), \rho_r][I_q, f_q] = [R^1N(R), \rho_{(r)fq}] \neq 0$ and $[R^1N(R), p_r][I_{q'}, f_{q'}] = [R^1N(R), p_r f_{q'}] \in R$.

(ii) Let us consider $q_j \in Q$, for $j = 1, 2, 3$, and take $a \in A_{(q_1q_2)q_3} \cap A_{q_1(q_2q_3)}$. By definition of $A_{q_1(q_2q_3)}$ we have $(a)\, f_{q_1} \in I_{q_2q_3}$; therefore there exist $y_i \in R^1$ and $a_i \in A_{q_2q_3}$ such that $(a)\, f_{q_1} = \sum y_i a_i$. Then for every $x \in R^1$,

$$\begin{aligned}
(xa)f_{(q_1q_2)q_3} &= x((a)f_{q_1}f_{q_2})f_{q_3}\\
&= x(((a)f_{q_1})f_{q_2})f_{q_3}\\
&= \sum x((y_ia_i)f_{q_2})f_{q_3}\\
&= \sum x((y_i)((a_i)f_{q_2}))f_{q_3}\\
(xa)f_{q_1(q_2q_3)} &= x((a)f_{q_1})f_{q_2}f_{q_3}\\
&= x(\sum((y_ia_i)f_{q_2}f_{q_3}\\
&= \sum x(y_i((a_i)f_{q_2})f_{q_3}.
\end{aligned}$$

So, if $[I_{q_1}, f_{q_1}] \in N(R)$ (i.e., $f_{q_1} = p_\lambda$ with $\lambda \in N(R)$), then $(a)\, f_{q_1} = a\lambda \in N(I_{q_2q_3})$ and therefore $(xa)\ f_{(q_1q_2)q_3} = x((a\lambda)\ f_{q_2})\, f_{q_3} = (xa)\ f_{(q_1q_2)q_3}$; if $[I_{q_2}, f_{q_2}] \in N(R)$, then $y_i((a_i)\ f_{q_2})\, f_{q_3} = (y_i((a_i)\ f_{q_2})\, f_{q_3}$. So, in any case, $(xa)\ f_{q_1(q_2q_3)} = (xa)\, f_{q_1(q_2q_3)}$, which implies that $N(R) \subset N(Q)$.

(iii) For every $p_1 r \in R$ and $q \in Q$ we have $(q, r, r) = 0$ and $S(p, q, r) = 0$. (2.5) Otherwise, if there exists $q \in Q$ such that $(q, r, r) \neq 0$, let us consider $\lambda \in N(R)$ such that $q\lambda \in R$ and $0 \neq (q, r, r)\lambda = (q, r, \lambda r) = (q, r, [r, \lambda]) + (q, r, r\lambda) = 0$ (by (ii) since $[r, \lambda] \in N(R)$), a contradiction.

Now, again if there exits $q \in Q$ such that $S(\rho, q, r) \neq 0$, let us consider $\lambda \in N(R)$ such that $\lambda q \in R$ and $0 \neq \lambda S(\rho, q, r) = S(\lambda\rho, q, r) = S([\lambda, \rho], q, r) + S(\rho, \lambda q, r) = 0$(by Teichmüller identity and (ii) since $[\lambda,\rho] \in N(R)$), a contradiction.

Now, given $\lambda \in N(R)$ and $q \in Q$, there exists $\mu \in N(R)$ such that $\mu[\lambda, q] \in R$. So for every $r, s\ \lambda \in R$ we have

$(r, \mu\lambda q, s) = -(r, s, \mu\lambda q)$ from (2.1)
$= -\mu(r, s, \lambda q)$ as the ring is Weakly Novikov
$= -\mu(r, s, q\lambda)$ as $[q,\lambda] \subseteq N(R)$
$= -(r, s, \mu q\lambda)$ by Weakly Novikov identity
$= (r, \mu q\lambda, s)$ by (2.1)

So we have $\mu[\lambda, q] \in N(R)$ for every $q \in Q$, $\lambda \in N(R)$ and $\mu \in A_{[\lambda, q]}$. Now, if there exist $p, p' \in Q$ such that $([\lambda, q], p, p') \neq 0$, then there exists $\mu \in N(R)$ such that $\mu[\lambda, q] \in R$ and $0 \neq \mu([\lambda, q], p, p') = (\mu[\lambda, q], p, p') = 0$, by (ii) and (2.5), a contradiction. Moreover, if $(p, ([\lambda, q], p') \neq 0$ there exists $\mu' \in N(R)$ such that $\mu' p[\lambda, q] \in R$ and $0 \neq \mu(\mu' p, p', [\lambda, q]) = (\mu' p, p'\mu [\lambda, q]) = -(\mu' p, \mu [\lambda, q], p') = 0$ by Weakly Novikov identity and from (2.1), a contradiction. From the above argument we also see that $(p, p', [\lambda, q]) = 0$ which implies the $[\lambda, q] \in N(Q)$.

(iv) Let $p_1, p_2, p_3 \in Q$ and $p = (p_1, p_2, p_3) + (p_1, p_3, p_2) \neq 0$. In view of (ii), (iii), Teichmuller identitiy, Weakly Novikov identity and (2.1), for any $\lambda \in N(R)$ we have $\lambda(p_1, p_2, p_3) = (\lambda p_1, p_2, p_3) = (p_1\lambda, p_2, p_3) = (p_1, \lambda p_2, p_3) - (p_1, \lambda, p_2 p_3) + p_1(\lambda, p_2, p_3) + (p_1\lambda, p_2) p_3 = (p_1, \lambda p_2, p_3) = -(p_1, p_3, \lambda p_2) = -\lambda(p_1, p_3, p_2) = \lambda(p_1, p_2, p_3) = (p_1, p_2, \lambda p_3)$.

By (i) and (ii), there exists $\lambda_3 \in N(R) \subseteq N(Q)$, such that $\lambda_3 p \neq 0$ and $\lambda_3 p_3 \in R$. Similarly, there exists $\lambda_2 \in N(R) \subseteq N(Q)$, such that $\lambda_2\lambda_3 p \neq 0$ and $\lambda_2 p_2 \in R$. Finally, there exists

$\lambda_1 \in N(R) \subseteq N(Q)$, such that $\lambda_1\lambda_2\lambda_3 p \neq 0$ and $\lambda_1 p_1 \in R$. Therefore $\lambda_1\lambda_2\lambda_3 p = (\lambda_1 p_1, \lambda_2 p_2, \lambda_3 p_3) + (\lambda_1 p_1, \lambda_3 p_3, \lambda_2 p_2) = 0$, since R satisfies (2.1). The contradiction proves that $p = 0$.

(v) Suppose that there exists $p, q \in$ Q such that $(q, p, p) \neq 0$. Then for every $\lambda \in Ap$, $(q, p, p)\lambda^2 = (q, p\lambda, p\lambda) = 0$ by (5).

Suppose first $A(R)$ is semiprime. We know that there exists $\alpha, \beta, \gamma \in N(R)$ such that $q\alpha, p\beta, p\gamma \in R$ and $(q\alpha, p\beta, p\gamma) \neq 0$. Now there exists $\lambda \in Ap$ such that $(q\alpha, p\beta, p\gamma)\lambda \neq 0$ and therefore, since A(R) is semiprime, $(q\alpha, p\beta, p\gamma)\lambda^2 \neq 0$ (otherwise the ideal generated by $(q\alpha, p\beta, p\gamma)\lambda$ will be nilpotent see [9] a contradiction.

Suppose now that $A(R)$ is 2 – torsion free. If $\lambda, \mu \in A_p$. $0 = (q, p, p)(\lambda + \mu)^2 = (q, p, p)\lambda^2 + (q, p, p)2\lambda\mu + (q, p, p)\mu^2 = 2(q, p, p)\lambda\mu$. Note that $(q, p, p)\lambda\mu = (q, p, p)\mu\lambda = (q, p, p)\mu\lambda$ so $(q, p, p)\lambda\mu = 0$ a contradiction. Now again suppose that there exist $p, q, r \in Q$ such that $S(p, q, r) \neq 0$. Applying the same argument as above we come across with a contradiction and hence we have Q is a (-1, 1) ring. ♦

Lemma 3.3: Let W be a left quotient ring of R and consider $q \in W$ then $(N(R):q) := \{\lambda \in N(R) / \lambda q \in R\} \in F^*$.

Proof: Given $0 \neq x \in R$ and $\mu \in N(R)$, there locusts $\gamma \in N(R)$ such that $\gamma x \neq 0$ and $\gamma(\mu q) \in R$; hence $\gamma\mu \in (N(R):q)$. ♦

Theorem 3.2: Let R be a (-1,1) ring such that $A(R)$ is 2-torsion free or semiprime .then R is a left quotient ring of itself if and only if the maximal left quotient ring R exists.

Proof : Let W be a left quotient ring of R .Given an element $q \in W$ by Lemma 3.1 and Lemma 3.2 $I_q ;= R^1(N(R))$; q)is a dense left ideal of R .Now following the proof of Theorem 3.1(i),the map $\phi : W \to Q$ defined by $\phi(q) := [I_q, \rho_q]$ is a monomorphism of (-1, 1) rings. ♦

Remark 3.1: By construction, the maximal left quotient ring is a (-1, 1) ring is unital with unit element $[R^1N(R), Id_R]$.

Examples of maximal left quotient rings are as follows:

Example 3.1: It is clear that the maximal left quotient ring of an associative ring is its maximal left quotient ring as a (-1, 1) ring.

Example 3.2: Let Q be a Cayley-Dickson algebra over its center. Then $Q_{max}{}^l(Q) = Q$; Let W be a left quotient ring of Q, and take W in P. By hypothesis there exists $n \in Z(Q)$ (which is a field) such that $np \in Q$. so $p = n^{-1}(ap) \in Q$.

Example 3.3: If R is a Cayley – Dickson ring ,its maximal left quotient ring is a Cayley – Dickson algebra by definition R is a central order in a Cayley – Dickson algebra, denoted by Q. So Q is a left quotient ring of R ,which implies that $Q_{max}{}^l(R) = Q_{max}{}^l(Q) = Q$ by (3.2).

Example 3.4: Let us consider a family $\{R_\alpha\}$ of (-1, 1) rings such that for every α there exist the maximal left quotient ring of R_α, which we denote Q_α. Then $Q_{max}{}^l(\oplus R_\alpha)$ exists and is equal to $\prod Q_\alpha$, the direct product of Q_α : the Proof is analogous to Utumi (1956), definition of maximal Left quotient ring.

4. CLASSICAL LEFT QUOTIENT RINGS:

The next proposition, which is in Gomez Lozano and siles Molina (preprint), (5.7) and (6.7) (i), shows that the maximal ring of quotient gives us an appropriate framework in which to settle the different left quotient rings that have been investigated (Fountain – Gould and classical); see Essannouni and Kaidi (1994) and Gomezlozono and Siles Molina (preprint) for definitions.

This fact was used by Ahn and Marki to give a general theory of Fountain and Gould left order in the setting of associative rings.

Proposition 4.1: Let R be a (-1, 1) ring. If R is a classical (Fountain and Gould) left order in a (-1, 1) ring W, then W is a left quotient ring of R. So W is a subring of $Q_{max}^{l}(R)$.

Let us construct the classical left order of a left Ore (-1, 1) ring R as the subring of $Q_{max}^{l}(R)$ generated by R and the set $\{a^{-1} / a \in \text{Reg}(R) \cap N(R)\}$.

Let R be a (-1, 1) ring. We recall that R satisfies the left Ore condition relative to a nonempty set S if for every $a \in W$ and $x \in R$ there exist $b \in W$ and $y \in R$ such that $bx = ya$. We will say that R is left Ore if it verifies the left Ore condition relative to $Reg(R) \cap N(R) \neq \phi$, where $Reg(R)$ denoted the set of all regular elements of R.

Note that $Reg(R) \cap N(R) \neq 0$ implies that R is a left quotient ring of itself. So there exists the maximal left quotient ring of R, denoted by Q.

Lemma 4.2: Every element $a \in Reg(R) \cap N(R)$ is invertible in Q.

Proof: It is easy to prove that Ra is a dense left ideal of R. Moreover, the map h: $Ra \rightarrow R$, defined by $(xa)h = x$ for every $xa \in Ra$, belongs to $Hom^{*}_{M(R)}(R,R)$. Now $[Ra, h]$ is the inverse of a in Q. Furthermore, $[Ra, h] \in N(Q)$. ♦

Lemma 4.3 : (Common Denominator Theorem) . For every elements $a, b \in Reg(R) \cap N(R)$ there exist $c, d \in Reg(R) \cap N(R)$ such that $cb = da$. ♦

Theorem 4.4: Let R be a ring that satisfies the left Ore condition. Then $T = \{a^{-1}x / a \in \text{Reg}(R) \cap N(R), x \in R\}$ is a subring of Q such that R is a classical left order in T.

Proof: Given $a^{-1}x$, $b^{-1}y$, where $a, b \in Reg(R) \cap N(R)$ and $x, y \in R$, by common denominator Theorem there exists $c, d \in Reg(R) \cap N(R)$ such that $cb = da$. So $a^{-1}x + b^{-1}y = a^{-1}d^{-1}dx = b^{-1}c^{-1}cy = (da)^{-1}(dx + cy) \in T$. It is straight forward that $a^{-1}xb^{-1}y \in T$.

We now show that T is a (-1, 1) ring. Given $p, q, r \in T$, there exists $a, b, c \in Reg(R) \cap N(R)$ such that $ap, bq, cr \in R$, so $a^2b(q, p, p) = 0$ and $(abc)S(p, q, r) = 0$ and hence is a (-1, 1) ring. Now it is trivial that R is a classical left order in T. Also since $a^2b(q, p, p) = 0$ which implies from the above fact that $(p, p, q) = 0 = (q, p, p)$. Hence T can be seen as alternative too. ♦

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REFERENCES

[1] Anh. P. N., and Marki. L., left orders in regular rings with minimum condition for principal One-sided ideals, *Mathematical* Proceedings of the *Cambridge* Philosophical *Society,* **109**, 323-333 (1991). http://dx.doi.org/10.1017/S0305004100069784

[2] Anh. P. N., and Marki. L., A general theory of Fountain-Gould quotient rings, Mathematica Slovaca, **44(2)**, 225-235 (1994).

[3] Beidar. K. L., and A.V.Mikhalov. A. V., Structure of nondegenarate alternative algebras, Journal of Soviet Mathematics, **47(3)**, 2525-2536 (1989). http://dx.doi.org/10.1007/BF01102996

[4] Castellon Sersano. A., Ternandez Lopez. A., Garcia Martin. A., and Martin Gonzalez. C., Strongly prime alternative pairs with minimal inner ideals, Manuscript Mathematica, **90**, 479-487 (1996). http://dx.doi.org/10.1007/BF02568320

[5] Essannouni. H., and Kaidi. A., Goldie's theorem for alternative rings, Proceedings of American Mathematical Society, **1328**, 39-45(1994). http://dx.doi.org/10.1090/S0002-9939-1994-1181162-5

[6] Fountain. J., and Gould. V., Orders in rings without identity, Communication in Algebra, **18(7)**, 3085-3110 (1990). http://dx.doi.org/10.1080/00927879008824062

[7] Fernandez Lopez. A., and Garcia Rus. E., Algebraic Lattices and nonassociative structures, Proceedings of American Mathematical Socciety, **126(11)**, 3211-3221 (1998). http://dx.doi.org/10.1090/S0002-9939-98-04443-8

[8] Fountain. J and Gould. V, Straight left orders in rings, Quarterly Journal Mathematics (oxford) **43** 303-311 (1992). http://dx.doi.org/10.1093/qmath/43.3.303

[9] Gomez Lozano. G., and Siles Molina. M., (preprint) Left quotient rings of alternative rings.

[10] Jayalakshmi. K., On (-1, 1) weakly Novikov rings Proceedings of International Smposium on Engineering And Natural Sciences (Macau) Vol. 1, Aug. (2013).

[11] Jayalakshmi. K., A note on (-1, 1) ring satisfying a.c.c, International Journal of Algebra and Statistics Vol.**3 (2)**, 34-41 (2014).

[12] Zhevlokov. K. A, Slinko. A. M, Shestakov. I. P., and Shirshov. A. I. (1982). Rings that are nearly associative, Academic Press, New York.