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On Eneström – Kakeya Theorem

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Abstract: In this paper we obtain some interesting Eneström-Kakeya type theorems concerning the location of zeros of polynomials. Our results extend and generalize Some well known results by putting less restrictive conditions on coefficients of polynomials.

Keywords and Phrases: Bounds, zeros, polynomial.
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1. INTRODUCTION AND STATEMENT OF RESULTS

The following elegant result which is well known in the theory of the distribution of the zeros of a polynomial is due to Eneström and Kakeya[6].

Theorem A: If $P(z)=a_nz^n+a_{n-1}z^{n-1}+....+a_1z+a_0$, is a polynomial of degree n,such that

$$a_n \geq a_{n-1} \geq .... \geq a_1 \geq a_0 > 0, \tag{1}$$

then all the zeros of P(z) lie in $|z|\leq 1$. This is a beautiful result but it is equally limited in scope as the hypothesis is very restrictive. In the literature [1,3,5,7,8], there exists some extensions and generalizations of Eneström-Kakeya Theorem.

Recently Aziz and Zargar[2], relaxed the hypothesis of Theorem A in several ways and proved the following results.

Theorem B: If $P(z)=a_nz^n+a_{n-1}z^{n-1}+....+a_1z+a_0$ is a polynomial of degree n such that for some $k\geq 1$.

$$ka_n \geq a_{n-1} \geq .... \geq a_1 \geq a_0 > 0 \tag{2}$$

then P(z) has all its zeros in $|z+k-1|\leq k$

Theorem C: If $P(z)=a_nz^n+a_{n-1}z^{n-1}+....+a_1z+a_0$ is a polynomial of degree $n\geq 2$, such that either $a_n \geq a_{n-2} \geq .... \geq a_3 \geq a_1 > 0$, and $a_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq a_0 > 0$, if n is odd

or
$a_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq a_0 > 0,$ and $a_{n-1} \geq a_{n-3} \geq .... \geq a_3 \geq a_1 > 0,$ if n is even, then all the zeros of P(z) lie in the circle

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq 1 + \frac{a_{n-1}}{a_n} \tag{3}$$

Theorem B is an interesting extension of Theorem A.

In this paper we shall first present the following extension of Theorem C analogous to Theorem B which among other things include Theorem A as a special case.

Theorem 1.1: If $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ is a polynomial of degree n≥2 such that for some k≥1, either $ka_n \geq a_{n-2} \geq .... \geq a_3 \geq a_1 > 0$ and $a_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq a_0 > 0$,if n is odd or $ka_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq a_0 > 0$ and $a_{n-1} \geq a_{n-3} \geq .... \geq a_3 \geq a_1 > 0$, if n is even then all the zeros of P(z) lie in the region

$$|z + \alpha||z + \beta| \leq (k + \frac{a_{n-1}}{a_n})$$

where α, β are the roots of the quadratic

$$z^2 + \frac{a_{n-1}}{a_n} z + k - 1 = 0 \tag{4}$$

Taking $a_{n-1} = 2a_n\sqrt{k-1}$ and noting that the quadratic $z^2+2\sqrt{(k-1)}z+k-1=0$ has two equal roots each is equal to $-\sqrt{(k-1)}$, we get the following:

Corollary 1: If $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ is a polynomial of degree n≥2 such that for some k≥1, either $ka_n \geq a_{n-2} \geq .... \geq a_3 \geq a_1 > 0$ and $2a_n\sqrt{k-1} = a_{n-3} \geq .... \geq a_2 \geq a_0 > 0,$ if n is odd or (5)

$ka_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq a_0 > 0$, and
$2a_n\sqrt{k-1} = a_{n-1} \geq a_{n-3} \geq .... \geq a_3 \geq a_1 > 0,$ if n is even.

then all the zeros of P(z) lie in the circle

$$\left| \mathrm{z} + \sqrt{\mathrm{k\text{-}1}} \right| \leq \left(k + 2\sqrt{k-1} \right)^{\frac{1}{2}} \tag{6}$$

Applying Corollary 1 to the polynomial

$$F(z) = b_{2n} z^{2n} + b_{2n-1} z^{2n-1} + ... + b_1 z + b_0,$$

of even degree 2n, we get

Corollary 2: if

$$F(z)=\sum_{j=0}^{2n} b_j z^j$$

is a polynomial of even degree 2n such that $kb_{2n} \geq b_{2n-2} \geq .... \geq b_2 \geq b_0 > 0,$ and $(k-1)b_{2n} = b_{2n-1} \geq b_{3n-3} \geq .... \geq b_3 \geq b_1 > 0$, then all the zeros of P(z) lie in

$$\left|z+\sqrt{\text{k-1}}\right| \leq \left(k+2\sqrt{k-1}\right)^{\frac{1}{2}}$$

Remark 1: Corollary 2 includes Eneström-Kakeya Theorem (Theorem A) as a special case. To see that we take k=1 in corollary 2 and

$$b_{2n-1} = b_{3n-3} = ... = b_3 = b_1 = 0,$$

it follows that if $b_{2n} \geq b_{2n-2} \geq .... \geq b_2 \geq b_0 > 0$, then all the zeros of

$$F(z) = b_{2n}z^{2n} + b_{2n-2}z^{2n-2} + ... + b_2 z^2 + b_0$$
$$= b_{2n}(z^2)^n + b_{2n-2}(z^2)^{n-1} + ... + b_2(z^2) + b_0$$

lie in $|z| \leq 1$. Replacing z^2 by z and b_{2j} *by* b_j j = 0,1,2....,n it follows that if

$$a_n \geq a_{n-1} \geq .... \geq a_1 \geq a_0 > 0$$

then all the zeros of

$$P(z)=\sum_{j=0}^{n} a_j z^j$$

lie in $|z| \leq 1$. which is precisely the conclusion of Eneström-Kakeya Theorem.
Taking k=2, in corollary 1 the following result follows ;

Corollary 3: if

$$P(z)=\sum_{j=0}^{n} a_j z^j$$

is a polynomial of degree $n \geq 2$ such that either
$2a_n \geq a_{n-2} \geq .... \geq a_3 \geq a_1 > 0$ and $2a_n = a_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq a_0 > 0$,
if n is odd
or
$2a_n = a_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq a_0 > 0$ and $2a_n \geq a_{n-2} \geq .... \geq a_2 \geq a_0 > 0$,
if n is even, then all the zeros of P(z) lie in

$$|z+1| \leq 2.$$

Next we prove the following generalization of Theorem C

Theorem 1.2: if

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+....+a_{2\lambda}z^{2\lambda}+.....+a_1z+a_0,$$

is a polynomial of degree n≥2 such that either

$$a_n\geq a_{n-2}\geq.......\geq a_{2\lambda+1}\leq a_{2\lambda-1}\leq...\leq a_3\leq a_1>0$$

and $a_{n-1}\geq a_{n-3}\geq.......\geq a_{2\lambda}\leq a_{2\lambda-2}\leq...\leq a_2\leq a_0>0$, for some integer $\lambda,\ 0\leq\lambda\leq\dfrac{n-1}{2}$, if n is odd, or

$a_n\geq a_{n-2}\geq.......\geq a_{2\lambda}\leq a_{2\lambda-2}\leq...\leq a_2\leq a_0>0$, and

$a_{n-1}\geq a_{n-3}\geq.......\geq a_{2\lambda+1}\leq a_{2\lambda-1}\leq...\leq a_3\leq a_1>0,$ for some integer $\lambda,\ 0\leq\lambda\leq\dfrac{n-2}{2}$ if n is even then all the zeroes of P(z) lie in the closed disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\leq 1+\frac{a_{n-1}+2(a_0+a_1-(a_{2\lambda}+a_{2\lambda+1}))}{a_n} \tag{7}$$

The following result is obtained by applying Theorem 1.2 to the polynomial P(tz):

Corollary 4: If

$$P(z)=\sum_{j=0}^{n}a_jz^j$$

is a polynomial of degree n≥2 such that for some t>0 either

$a_nt^n\geq a_{n-2}t^{n-2}\geq.......\geq a_{2\lambda+1}t^{2\lambda+1}\leq a_{2\lambda-1}t^{2\lambda-1}\leq...\leq a_3t^3\leq a_1t>0,$ and

$a_{n-1}t^{n-1}\geq a_{n-3}t^{n-3}\geq.......\geq a_{2\lambda}t^{2\lambda}\leq a_{2\lambda-2}t^{2\lambda-2}\leq...\leq a_2t^2\leq a_0>0,$ for some integer $\lambda,\ 0\leq\lambda\leq\dfrac{n-1}{2}$, if n is odd

$a_nt^n\geq a_{n-2}t^{n-2}\geq.......\geq a_{2\lambda}t^{2\lambda}\leq a_{2\lambda-2}t^{2\lambda-2}\leq...\leq a_2t^2\leq a_0>0,$ and

$a_{n-1}t^{n-1}\geq a_{n-3}t^{n-3}\geq.......\geq a_{2\lambda+1}t^{2\lambda+1}\leq a_{2\lambda-1}t^{2\lambda-1}\leq...\leq a_3t^3\leq a_1t>0,$ for some integer $\lambda,\ 0\leq\lambda\leq\dfrac{n-2}{2}$, if n is even ,then all the zeros of P(z) lie in the closed disk

$$\left|z+\frac{a_{n-1}}{a_n}\right|\leq t+\frac{t^{n-1}a_{n-1}+2(a_0+a_1t-t^{2\lambda}(a_{2\lambda}+a_{2\lambda+1}t))}{t^{n-1}a_n} \tag{8}$$

2. PROOFS OF THE THEOREMS

Proof of Theorem 1.1: consider

$$
\begin{aligned}
F(z) &= (1-z^2)P(z)\\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + (a_n - a_{n-2})z^n + \ldots + (a_3 - a_1)z^3\\
&\quad + (a_2 - a_0)z^2 + a_1 z + a_0,
\end{aligned}
$$

For |z|>1, we have

$$
\begin{aligned}
|F(z)| &= \left| -a_n z^{n+2} - a_{n-1}z^{n+1} - ka_n z^n + a_n z^n + (ka_n - a_{n-2})z^n \right.\\
&\quad \left. + \ldots + (a_3 - a_1)z^3 + (a_2 - a_0)z^2 + a_1 z + a_0 \right|\\
&\geq |z|^n \left\{ \left| a_n z^2 + a_{n-1} z + (k-1)a_n \right| \right.\\
&\quad \left. - \left| (ka_n - a_{n-2}) + (a_{n-1} - a_{n-3})\frac{1}{z} + \ldots + (a_3 - a_1)\frac{1}{z^{n-3}} + (a_2 - a_0)\frac{1}{z^{n-2}} + a_1 \frac{1}{z^{n-1}} + a_0 \frac{1}{z^n} \right| \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \left| z^2 + \frac{a_{n-1}}{a_n} z + (k-1) \right|\\
&\quad - \frac{1}{|a_n|} \left\{ (ka_n - a_{n-2}) + (a_{n-1} - a_{n-3})\frac{1}{|z|} + \ldots + (a_3 - a_1)\frac{1}{|z|^{n-3}} + (a_2 - a_0)\frac{1}{|z|^{n-2}} + a_1 \frac{1}{|z|^{n-1}} + a_0 \frac{1}{|z|^n} \right\}
\end{aligned}
$$

$$
> \left| z^2 + \frac{a_{n-1}}{a_n} z + (k-1) \right| - \left(k + \frac{a_{n-1}}{a_n}\right)
$$

>0,if

$$
\left| z^2 + \frac{a_{n-1}}{a_n} z + (k-1) \right| > \left(k + \frac{a_{n-1}}{a_n}\right)
$$

Hence all the zeros of F(z) whose modulus is greater than 1 lie in the region

$$\left|z^2+\frac{a_{n-1}}{a_n}z+(k-1)\right|\le(k+\frac{a_{n-1}}{a_n}) \tag{9}$$

But those zeros of F(z) whose modulus is less than or equal to 1 already satisfy the inequality(9). Since all the zeros of P(z) are also the zeros of F(z), therefore it follows that all the zeros of P(z) lie in the region(9).

Let α and β be the roots of the quadratic $z^2+\frac{a_{n-1}}{a_n}z+(k-1)=0$, therefore from(9), we have $|z-\alpha||z-\beta|\le k++\frac{a_{n-1}}{a_n}$. which completes the proof of Theorem 1.1

Proof of Theorem 1.2: Consider

$$\begin{aligned}F(z) &= (1-z^2)P(z)\\ &=-a_nz^{n+2}-a_{n-1}z^{n+1}+(a_n-a_{n-2})z^n+\ldots+(a_3-a_1)z^3\\ &+(a_2-a_0)z^2+a_1z+a_0,\end{aligned}$$

therefore for |z|>1, we have

$$\begin{aligned}|F(z)|&=\left|-(a_nz^n+a_{n-1})z^{n+1}+(a_n-a_{n-2})z^n+\ldots\right.\\ &\left.+(a_3-a_1)z^3+(a_2-a_0)z^2+a_1z+a_0\right|\\ &\ge\left|z^{n+1}\right|\left\{\left|a_nz+a_{n-1}\right|-(\left|a_n-a_{n-2}\right|\frac{1}{|z|}+\ldots\right.\\ &\left.+\left|a_3-a_1\right|\frac{1}{\left|z^{n-2}\right|}+\left|a_2-a_0\right|\frac{1}{\left|z^{n-1}\right|}+\left|a_1\right|\frac{1}{\left|z^n\right|}+\frac{|a_0|}{\left|z^{n+1}\right|})\right\}\\ &>\left|z^{n+1}\right|\{\left|a_nz+a_{n-1}\right|-(\left|a_n-a_{n-2}\right|+\ldots\\ &+\left|a_3-a_1\right|+\left|a_2-a_0\right|+\left|a_1\right|+\left|a_0\right|)\}\\ &=\left|z^{n+1}\right|\left\{\left|a_nz+a_{n-1}\right|-(\sum_{j=0}^{n}\left|a_j-a_{j-2}\right|+\left|a_1\right|+\left|a_0\right|)\right\}\\ &=\left|z^{n+1}\right|\left\{\left|a_nz+a_{n-1}\right|\right.\\ &\left.-(a_0+a_1+\sum_{k=1}^{\frac{n}{2}}\left|a_{2k}-a_{2k-2}\right|+\sum_{k=1}^{\frac{n-k}{2}}\left|a_{2k+1}-a_{2k-1}\right|)\right\}\end{aligned} \tag{10}$$

Assuming first that n is even then from (10), for |z|=1, we have

$$|F(z)| > |z^{n+1}| \Bigg\{ |a_n z + a_{n-1}| - (a_0 + a_1 + \sum_{k=1}^{\lambda} |a_{2k-2} - a_{2k}|$$

$$+ \sum_{k=\lambda+1}^{\frac{n}{2}} |a_{2k} - a_{2k-2}| + \sum_{k=\lambda+1}^{\frac{n-2}{2}} |a_{2k+1} - a_{2k-1}|) \Bigg\}$$

$$= |z^{n+1}| \{ |a_n z + a_{n-1}| - 2(a_0 + a_1 - a_{2\lambda} - a_{2\lambda+1}) + a_n + a_{n-1} \}$$

>0,if

$$\left| z + \frac{a_{n-1}}{a_n} \right| > 1 + \frac{a_{n-1} + 2(a_0 + a_1 - a_{2\lambda} - a_{2\lambda+1})}{a_n} \tag{11}$$

In case n is odd it can be easily seen that |P(z)|>0 if (11) holds. Hence all those zeros of P(z) whose modulus is greater than 1 lie in the circle

$$\left| z + \frac{a_{n-1}}{a_n} \right| > 1 + \frac{a_{n-1} + 2(a_0 + a_1 - a_{2\lambda} - a_{2\lambda+1})}{a_n} \tag{12}$$

But all those zeros of P(z) whose modulus is less than or equal to 1 already satisfy (12). Therefore it follows that all the zeros of P(z) lie in the circle(12). which proves Theorem(1.2).

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