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On The Maximum Modulus of a Polynomial

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Abstract: Let $P(z)$ be a polynomial of degree n not vanishing in $|z| < k$ where $k >= 1$. It is known that

$$Max_{|z|=R>1}|P(z)| \leq \frac{\left(R+k\right)^n}{\left(R+k\right)^n+\left(1+Rk\right)^n} \left\{\left(R^n+1\right)Max_{|z|=1}|P(z)| - \left(R^n - \left(\frac{1+Rk}{R+k}\right)^n\right)Min_{|z|=k}|P(z)|\right\}.$$

In this paper, we obtain a refinement of this and many other related results.

Key words and Phrases: Polynomial, Zeros, Maximum modulus.
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1. INTRODUCTION AND STATEMENT OF RESULTS

For an arbitrary entire function, let $M(f,r) = Max_{|z|=r}|f(z)|$ and $m(f,r) = Min_{|z|=r}|f(z)|$. Let $P(z)$ be a polynomial of degree n, then

$$M(P,R) \leq R^n M(P,1), \quad R \geq 1. \tag{1}$$

Inequality (1) is a simple deduction from Maximum Modulus Principle (see [6], p-442). It was shown by Ankeny and Rivilin [1] that if $P(z)$ does not vanish in $|z| < 1$, then (1) can be replaced by

$$M(P,R) \leq \frac{R^n+1}{2} M(P,1), \quad R \geq 1. \tag{2}$$

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The bound in (2) was further improved by Aziz and Dawood [2], who under the same hypothesis proved

$$M(P,R)\le\frac{R^n+1}{2}M(P,1)-\frac{R^n-1}{2}m(P,1),\ R\ge 1. \tag{3}$$

As a generalization of (2), Aziz and Mohammad [3] proved that if $P(z)\neq 0$ in $|z|<k,\ k\ge 1$, then for $R\ge 1$,

$$M(P,R)\le\frac{(R^n+1)(R+k)^n}{(R+k)^n+(1+Rk)^n}M(P,1), \tag{4}$$

whereas under the same hypothesis, Aziz and Zargar [4] extended inequality (3) by showing that

$$M(P,R)\le\frac{(R+k)^n}{(R+k)^n+(1+Rk)^n}\left\{(R^n+1)M(P,1)-\left(R^n-\left(\frac{1+Rk}{R+k}\right)^n\right)m(P,k)\right\}. \tag{5}$$

In this note, we obtain a refinement of (5) and hence of inequalities (2), (3) and (4) as well. More precisely, we prove

Theorem 1. If $P(z)$ is a polynomial of degree $n\ge 3$ which does not vanish in $|z|<k, k\ge 1$, then for $R\ge 1$

$$M(P,R)\le\frac{(R+k)^n}{(R+k)^n+(1+Rk)^n}\Bigg\{(R^n+1)M(P,1) - \big||P'(0)|-|Q'(0)|\big|\left(\frac{R^n-1}{n}-\frac{R^{n-2}-1}{n-2}\right) - \left(R^n-\left(\frac{1+Rk}{R+k}\right)^n\right)m(P,k)\Bigg\}, \tag{6}$$

where $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$.

Remark: Since for $R>1, \frac{R^x-1}{x}$ is an increasing function of *x*, the expression $\big||\mathrm{P}'(0)|-|\mathrm{Q}'(0)|\big|\left(\frac{R^n-1}{n}-\frac{R^{n-2}-1}{n-2}\right)$ is always non-negative. Thus for polynomials of degree greater than two, our theorem sharpens the bound obtained in (5). (The cases when polynomial *P*(*z*) is of degree 1 or 2 is uninteresting because then *M*(*P,R*) can be calculated trivially). In fact, excepting the case when $P'(0)=Q'(0)$, the bound obtained by our theorem is always sharp than the bound that is obtained in (5).

2. LEMMAS

For the proof of Theorem 1, we require the following lemmas.

Lemma 1. If *P*(*z*) is a polynomial of degree $n \geq 3$ and $Q(z)=z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, then for every R ≥ 1 and $0 \leq \theta < 2\pi$,

$$|P(Re^{i\theta})|+|Q(Re^{i\theta})| \leq (R^n+1)M(P,1) - \left(\frac{R^n-1}{n}-\frac{R^{n-2}-1}{n-2}\right)\left||P'(0)|-|Q'(0)|\right|. \tag{7}$$

The above lemma is due to Jain [5].

Lemma 2. If *P*(*z*) is a polynomial of degree *n* which does not vanish in $|z|<k,\ k>0$, then for every $R \geq 1, r \leq k$ and for every $\theta, 0 \leq \theta < 2\pi$,

$$\left|P(R\,r\,e^{i\theta})\right| \leq \left(\frac{Rr+k}{r+Rk}\right)^n \left|R^n P\left(\frac{re^{i\theta}}{R}\right)\right| - \left\{\left(\frac{Rr+k}{r+Rk}\right)^n R^n-1\right\} m(p,k). \tag{8}$$

The above lemma is due to Aziz and Zargar [4].

3. PROOF OF THE THEOREM

Proof of the Theorem 1. Since $P(z) \neq 0$ in $|z|<k,\ k \geq 1$, using Lemma 2, it follows from (8) with *r* = 1, that

$$\left|P(Re^{i\theta})\right| \leq \left(\frac{R+k}{1+Rk}\right)^n \left|R^n P\left(\frac{e^{i\theta}}{R}\right)\right| - \left\{\left(\frac{R+k}{1+Rk}\right)^n R^n-1\right\} m(p,k) \tag{9}$$

for every θ, $0 \leq \theta < 2\pi$ and R ≥ 1. Since $Q(z) = z^n \overline{P\left(\frac{1}{z}\right)}$, we have

$$\left|Q(\mathrm{Re}^{i\theta})\right| = \left|R^n P\left(\frac{e^{i\theta}}{R}\right)\right|. \tag{10}$$

Using (10) in (9), we get

$$\left|P\left(\mathrm{Re}^{i\theta}\right)\right| \leq \left(\frac{R+k}{1+Rk}\right)^n \left|Q\left(\mathrm{Re}^{i\theta}\right)\right| - \left\{\left(\frac{R+k}{1+Rk}\right)^n R^n - 1\right\} m(p,k).$$

This implies,

$$\begin{aligned}\frac{(R+k)^n + (1+Rk)^n}{(1+Rk)^n}\left|P\left(\mathrm{Re}^{i\theta}\right)\right| \leq \left(\frac{R+k}{1+Rk}\right)^n \left\{\left|P\left(\mathrm{Re}^{i\theta}\right)\right| + \left|Q\left(\mathrm{Re}^{i\theta}\right)\right|\right\} \\ - \left\{\left(\frac{R+k}{1+Rk}\right)^n R^n - 1\right\} m(p,k).\end{aligned} \tag{11}$$

Inequality (11) yields with the help of Lemma 1 that

$$\begin{aligned}\frac{(R+k)^n + (1+Rk)^n}{(1+Rk)^n}\left|P\left(\mathrm{Re}^{i\theta}\right)\right| \leq \left(\frac{R+k}{1+Rk}\right)^n \left\{\left(R^n + 1\right) M(P,1)\right. \\ \left. - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right)\left\|P'(0)\right| - \left|Q'(0)\right\|\right\} \\ - \left\{\left(\frac{R+k}{1+Rk}\right)^n R^n - 1\right\} m(p,k).\end{aligned} \tag{12}$$

From (12), it follows that for every $\theta, 0 \leq \theta < 2\pi$ and $R \geq 1$,

$$\begin{aligned}\left|P(\mathrm{Re}^{i\theta}\right| \leq \frac{(R+k)^n}{(R+k)^n + (1+Rk)^n}\left\{\left(R^n + 1\right) M(P,1)\right. \\ \left. - \|P'(0)| - |Q'(0)\|\left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right) - \left(R^n - \left(\frac{1+Rk}{R+k}\right)^n\right) m(P,k)\right\},\end{aligned}$$

which is equivalent to the desired result and this completes the proof of Theorem 1.

REFERENCES

[1] Ankeny, N.C. and Rivilin, T.J., *On a theorem of S. Bernstein*, Pacific J. Math.,**5**(1955), 849-852. http://dx.doi.org/10.2140/pjm.1955.5.849

[2] Aziz, A. and Dawood, Q.M., *Inequalities for a polynomial and its derivative*, J. Approx. Theory, **54** (1988), 306-313. http://dx.doi.org/10.1016/0021-9045(88)90006-8

[3] Aziz, A. and Mohammad, Q.G., *Growth of polynomials with zeros outside a circle*, Proc. Amer. Math. Soc, **81** (1981), 549-553. http://dx.doi.org/10.1090/S0002-9939-1981-0601727-7

[4] Aziz, A. and Zargar, B.A., *Growth of maximum modulus of polynomials with prescribed zeros*, Glas. Mate., **37** (2002), 73-81.

[5] Jain, V.K., *On maximum mmodulus of polynomials*, Annales Univ. Marie Curie-SKlodowska Lublin-Polonia, LV, **7**(2001), 81-84.

[6] Milovanovic, G.V., Mitrivonic, D.S., and Rassias, Th.M. *Topics in Polynomials, Extremal properties, Inequalities, Zeros*, World Scientific, Publishing Co., Singapore, 1994. http://dx.doi.org/10.1142/9789814360463