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Certain Characterizations of Tight Gabor Frames on Local Fields

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Abstract: Gabor systems are generated by modulations and translations of a single function. Many researchers studied Gabor frames in Hilbert spaces. The concepts Gabor frames on local fields, first introduced by Li and Jiang. They studied the existence of a Gabor frame on local fields and also established some necessary conditions and two sufficient conditions of Gabor frame for local fields. Inspired by above paper, in this paper, we study certain characterizations of tight Gabor frames on the local fields of positive characteristic.

Keywords: Frame, local field, Gabor frames.

1. INTRODUCTION

The concept of frames for Hilbert spaces, first introduced by Duffin and Schaeffer [6]. The idea of frame has been lost until Daubechies, Grossmann, and Meyer [7] brought attention to it in 1986. They showed that Duffin and Schaeffer's definition is an abstraction of a concept given by Gabor [8] in 1946 for doing signal analysis. Many researchers studied Gabor frames in $L^2\left(\mathbb{R}^d\right)$. There are many results for Gabor frame on $\mathbb{R}^d$ but the counterparts on local field are only studied by D. Li and H. Jiang [10]. They investigated the existence of a Gabor frame $\left\{M_{u(m)b}T_{u(n)a}g\right\}_{m,n\in\mathbb{P}}$ on local fields and also established some necessary conditions and two sufficient conditions of Gabor frame for $L^2\left(K\right)$.

Gabor system is a particular case of wave packet systems. Wave packet system has been studied by many researchers in different setups. Recently, Abdullah and Shah [1] have established a necessary and sufficient condition for the wave packet system $\left\{D_{p^j}T_{u(n)a}E_{u(m)b}\psi\right\}_{j\in Z,m,n\in\mathbb{P}}$ to be a frame for $L^2\left(K\right)$

and Shah and Abdullah [12] studied a necessary condition for the existence of wave packet frame for $L^2(\mathbb{R})$, whereas Abdullah has given the characterization of all tight wave packet frames for $L^2(\mathbb{R})$ and $H^2(\mathbb{R})$ by imposing some mild condition on the generator in [2].

Inspired by the work of D. Li and H. Jiang [10], in this paper, we study certain characterizations of tight Gabor frames on the local fields. The paper is structured as follows. In Section 2, we introduce some notations and preliminaries related to local fields to be used throughout the paper and some results which we will use in the proof of main results. In Section 3, we prove the characterizations of Gabor frames on local fields of positive characteristic.

2. PRELIMINARIES

In this section, we list some notations of local fields to be used throughout the paper. For more details please refer to [9, 13].

A local field means an algebraic field and a topological space with the topological properties of locally compact, non-discrete, complete and totally disconnected, denoted by K. The additive and multiplicative groups of K are denoted by K^+ and K^*, respectively. We may choose a Haar measure dx for K^+. If $\alpha \neq 0, \alpha \in K$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the absolute value of α. We also let $|0| = 0$.

The map $x \to |x|$ has the following properties:

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|xy| = |x||y|$ for all $x, y \in K$;

(c) $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

The set $\mathcal{D} = \{x \in K : |x| \leq 1\}$ is called the ring of integers in K. Define $\mathcal{B} = \{x \in K : |x| < 1\}$. The set $\mathcal{B}$ is called the prime ideal in K. The prime ideal in K is the unique maximal ideal in $\mathcal{D}$. Let p be a fixed element of maximum absolute value in $\mathcal{B}$. Such an element is called a prime element of K. We have the fact that $|\mathcal{B}| = q^{-1}$ and $p = q^{-1}$. It follows that if $x \neq 0$ and $x \in K$, then $|x| = q^k$ for some $k \in \mathbb{Z}$.

Let $\mathcal{B}^k = p^k \mathcal{D}\{x \in K : |x| \leq q^{-k}\}, k \in \mathbb{Z}$. Each $\mathcal{B}^k$ is compact subgroup of K^+. $\mathcal{D} = \mathcal{B}^0$ is a ring of integers in K. So, $|\mathcal{D}| = 1$ and $|\mathcal{B}^k| = q^{-k}$. χ is a fixed character on K^+ that is trivial on $\mathcal{D}$ but is non-trivial on $\mathcal{B}^{-1}$. It follows that χ is constant on cosets of $\mathcal{D}$ and that if $y \in \mathcal{B}^k$, then $\chi_y\left(\chi_y(x) = \chi(yx)\right)$ is constant on cosets of $\mathcal{B}^{-k}$.

We now proceed to impose a "natural" order on the sequence $\{u(n)\}_{n=0}^{\infty}$.

We recall $\mathcal{B}$ is the prime ideal in $\mathcal{D}, \mathcal{D}/\mathcal{B} \cong GF(q) = \Gamma, q = p^c, p$ a prime, c a positive integer and $\rho: \mathcal{D} \to \Gamma$ the canonical homomorphism of $\mathcal{D}$ on to Γ. Note that $\Gamma = GF(q)$ is a c-dimensional vector space over $GF(\rho) \subset \Gamma$. We choose a set $\{1 = \varepsilon_0, \ldots, \varepsilon_{c-1}\} \subset \mathcal{D}^*$ such that $\{\rho(\varepsilon_k)\}_{k=0}^{c-1}$ is a basis of $GF(q)$ over $GF(q)$.

Definition 2.1. For $n, 0 \le n < q, n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1},\ 0 \le a_k < p$ and $k = 0, \ldots, c-1$, we define

$$u(n) = (a_0 + a_1\varepsilon_1 + \cdots + a_{c-1}\varepsilon_{c-1}) p^{-1},\ \left(0 \le n < q\right). \tag{2.1}$$

For $n = b_0 + b_1 q + \cdots + b_s q^s, 0 \le b_k < q, n \ge 0,$ we set

$u(n) = u(b_o) + p^{-1} u(b_1) + \cdots + p^{-s} u(b_s).$

Note that for $n, m \ge 0$ in general, it is not true that $u(n+m) = u(n) + u(m)$. However, it is true for all $r, k \ge 0,\ u(rq^k) = p^{-k} u(r)$, and for $r, k \ge 0,\ 0 \le t < q^k$, $u\left(rq^k + t\right) = u\left(rq^k\right) + u(t) = p^{-k} u(r) + u(t)$.

Hereafter we will denote $\chi_{u(n)}$ by $\chi_n (n \ge 0)$. We also often use the following number set throughout this paper: $\mathbb{P} = \{0, 1, 2, \ldots\}$.

Definition 2.2. A function f defined on K is said to be periodic with period a if $f\left(x + u(l)a\right) = f(x)$ for all $x \in K$ and $l \in \mathbb{P}$.

Definition 2.3. [3, 5]. A sequence $\{f_k\}_{k=1}^{\infty}$ in Hilbert space $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants A, B with $0 < A \le B < \infty$ satisfying

$$A\|f\|^2 \le \sum_{k=1}^{\infty} |< f, f_k >|^2 \le B\|f\|^2, \forall f \in \mathrm{H}. \tag{2.2}$$

If only the right hand side inequality holds, we say that $\{f_k\}_{k=1}^{\infty}$ is a Bessel system with constant B. A frame is a tight frame if A and B can be chosen so that $A = B$ and is a normalized tight frame (NTF) if $A = B = 1$.

Definition 2.4. Let a and b be two given positive real numbers. For any fixed function $g \in L^2(K)$, the family of functions of the form

$$M_{u(m)b} T_{u(n)a} g = \chi_m(bx) g(x - u(n)a),\ m, n \in \mathbb{P},\quad x \in K \tag{2.3}$$

is called a Gabor frame for $L^2(K)$ if there exist constants A and B, $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{m,n\in\mathrm{P}} |< f, M_{u(m)b}T_{u(n)a}g >|^2 \le B\|f\|^2, \forall f \in L^2(K), \tag{2.4}$$

where $M_{u(m)b}f(x) = \chi(u(m)bx)f(x)$ and $T_{u(n)a}f(x) = f(x-u(n)a)$ are the modulation and translation operators defined on $L^2(K)$, respectively.

Lemma 2.5. [10]. Let $f, g \in L^2(K), a, b \in K / \{0\}$ and $k \in \mathbb{P}$ be given. Then the series

$$\sum_{n\in\mathbb{P}} f(x-u(n)a)\overline{g(x-u(n)a-b^{-1}u(k))}, \quad x \in K \tag{2.5}$$

converges absolutely for a.e. $x \in K$, and it defines a function with period a, whose restriction to the set $G_a = \{x \in K; |x| \le |a|\}$ belongs to $L(G_a)$. In fact, $\sum_{n\in\mathbb{P}} \left| f(x-u(n)a)\overline{g(x-u(n)a-b^{-1}u(k))} \right| \in L(G_a)$.

Lemma 2.6 [10]. Let $f, g \in L^2(K), a, b \in K / \{0\}$ and $k \in \mathbb{P}$ be given. We consider the function $F_n \in L(G_{b^{-1}})$ defined by

$$F_n(x) = \sum_{k\in\mathbb{P}} f(x-b^{-1}u(k))\overline{g\left(x-u(n)a-b^{-1}u(k)\right)}.$$

Then, for any $m \in \mathbb{P}$.

$$< f, M_{u(m)b}T_{u(n)a}g > = \int_{G_b-1} F_n(x)\overline{\chi_m(bx)}\,dx.$$

In particular, the m-th Fourier coefficient of $F_n(x)$ with respect to the orthonormal basis $\left\{|b|^{\frac{1}{2}}\chi_m(bx)\right\}_{m\in\mathbb{P}}$ for $L^2(G_{b^{-1}})$ is $C_m = |b|^{\frac{1}{2}} < f, M_{u(m)b}T_{u(n)a}g >$.

Let $a, b \in K / \{0\}$ and $g \in L^2(K)$, we will often use the following functions defined by

$$G(x) = \sum_{n\in\mathbb{P}} \left|g\left(x-u(n)a\right)\right|^2, \; x \in K \tag{2.6}$$

$$H_k(x)=\sum_{n\in\mathbb{P}} g(x-u(n)a)\overline{g(x-u(n)a-b^{-1}u(k))},\ x\in K,\ k=0,1,\ldots \quad (2.7)$$

It is obvious that $G(x)$ and $H_k(x)$ are bounded functions with period a and $G(x)=H_0(x)$.

Lemma 2.7 [10]. Let $g\in L^2(K)$ and $a,b\in K/\{0\}$ be given. Suppose that f is a bounded measurable function with compact support. Then

$$\sum_{m,n\in\mathbb{P}} \left|< f, M_{u(m)b}T_{u(n)a}g >\right|^2 = \frac{1}{|b|}\int_K |f(x)|^2\ G(x)dx+\frac{1}{|b|}\sum_{k\in\mathbb{P}/\{0\}}\int_K \overline{f(x)}f(x-u(k)b^{-1})H_k(x)dx. \quad (2.8)$$

3. CHARACTERIZATIONS OF TIGHT GABOR FRAMES ON $L^2(K)$

Theorem 3.1. Let $f,g\in L^2(K)$ and $a,b\in K/\{0\}, n\in\mathbb{P}$ be given. Then
(i) f is orthogonal to $M_{u(m)b}T_{u(n)a}g$ for all $m\in\mathbb{P}/\{0\}$ if and only if there is a constants C so that

$$\sum_{k\in\mathbb{P}} f(x-b^{-1}u(k))\overline{g(x-u(n)a-b^{-1}u(k))}=C\ \ a.e.\ x\in K.$$

(ii) f is orthogonal to $M_{u(m)b}T_{u(n)a}g$ for all $m\in\mathbb{P}$ if and only if

$$\sum_{k\in\mathbb{P}} f(x-b^{-1}u(k))\overline{g(x-u(n)a-b^{-1}u(k))}=0\ a.e.\ x\in K.$$

Proof (i) Since f is orthogonal to $M_{u(m)b}T_{u(n)a}g$ for all $m\in\mathbb{P}/\{0\}$. Therefore

$$\left\langle f, M_{u(m)b}T_{u(n)a}g\right\rangle=0\ \forall m\in\mathbb{P}/\{0\}.$$

For any $m,n\in\mathbb{P}$; by Lemma 2.6, we have

$$\left\langle f, M_{u(m)b}T_{u(n)a}g\right\rangle=\int_{G_{b^{-1}}}\sum_{k\in\mathbb{P}} f(x-b^{-1}u(k))\overline{g(x-u(n)a-b^{-1}u(k))}\overline{\chi_m(bx)}\,dx.$$

Since $\left\{|b|^{\frac{1}{2}}\ \chi_m(bx)\right\}_{m\in\mathbb{P}}$ is an orthonormal basis for $L^2(G_{b^{-1}})$, it follows that for a given $n\in\mathbb{P}$, there exist a constant C, such that

$$\sum_{k\in\mathbb{P}} f(x-b^{-1}u(k))\overline{g(x-u(n)a-b^{-1}u(k))}=C,\ a.e.\ x\in G_{b^{-1}}.$$

Since $\sum_{k\in\mathbb{P}} f(x-b^{-1}u(k))\overline{g(x-u(n)a-b^{-1}u(k))}$ is $\dfrac{1}{b}$ -periodic. Therefore, we have

$$\sum_{k\in\mathbb{P}} f(x-b^{-1}u(k))\overline{g(x-u(n)a-b^{-1}u(k))}=C,\ a.e.\ x\in K.$$

This prove (i).

(ii) Since *f* is orthogonal to $M_{u(m)b}T_{u(n)a}g$ for all $m\in\mathbb{P}$. Therefore

$\left\langle f, M_{u(m)b}T_{u(n)a}g\right\rangle = 0\ \forall m\in\mathbb{P}\cdot$

Since $\left\{|b|^{\frac{1}{2}}\chi_m(bx)\right\}_{m\in\mathbb{P}}$ is an orthonormal basis for $L^2\left(G_{b^{-1}}\right)$, it follows that for a given $n\in\mathbb{P}$,

$$\left\langle f, M_{u(m)b}T_{u(n)a}g\right\rangle = 0\ \forall\ m\in\mathbb{P}$$

$$\Leftrightarrow \sum_{k\in\mathbb{P}} f(x-b^{-1}u(k))\overline{g(x-u(n)a-b^{-1}u(k))}=0,\ a.e.\ x\in G_{b^{-1}}.$$

Since $\sum_{k\in\mathbb{P}} f(x-b^{-1}u(k))\overline{g(x-u(n)a-b^{-1}u(k))}$ is $\dfrac{1}{b}$ -periodic. Therefore, we have

$$\sum_{k\in\mathbb{P}} f(x-b^{-1}u(k))\overline{g(x-u(n)a-b^{-1}u(k))}=0,\ a.e.\ x\in K.$$

This prove (ii).

Theorem 3.2. Let $g\in L^2(K)$ and $a,b\in K/\{0\}$. Then the following are equivalent:

(i) $\{M_{u(m)b}T_{u(n)a}g\}_{m,n\in\mathbb{P}}$ is a tight frame for $L^2(K)$ with frame bound A = 1.

(ii) We have

(a) $G(x)=\sum_{n\in\mathbb{P}}|g(x-u(n)a)|^2=|b|$, a.e.

(b) $G_k(x)=\sum_{n\in\mathbb{P}} g(x-u(n)a)\overline{g(x-u(n)a-b^{-1}u(k))}=0$ a.e. for all $k\neq 0$.

(iii) $g\perp M_{u(m)a^{-1}}T_{u(n)b^{-1}}g$ for all $(m,n)\neq(0,0)$ and $\|g\|^2=|ab|$.

(iv) $\left\{M_{u(m)a^{-1}}T_{u(n)b^{-1}}g\right\}_{m,n\in\mathbb{P}}$ is an orthonormal sequence and $\|g\|^2=|ab|$.

Moreover, when at least one of (i)-(iv) holds, $\{m_{u(m)b}T_{u(n)a}g\}_{m,n\in\mathbb{P}}$ is an orthonormal basis for $L^2(K)$ if and only if $\|g\|=1$.

Proof (i) $\Rightarrow$ (ii): Assume $\{M_{u(m)b}T_{u(n)a}g\}_{m,n\in\mathbb{P}}$ is a tight frame for $L^2(K)$ with frame bound A = 1. For any function $f\in L^2(K)$ which is supported on a ball Γ with diameter at most $|b|^{-1}$, we see that $\overline{f(x)}f\left(x-b^{-1}u(k)\right)=0$ for all $x\in K$ and for all $k\in\mathbb{P}/\{0\}$. By using Lemma 2.7, we have

$$\begin{aligned}\int_K|f(x)|^2\,dx &= \sum_{m,n\in\mathbb{P}}|<f,M_{u(m)b}T_{u(n)a}g>|^2\\ &=\frac{1}{|b|}\int_K|f(x)|^2\sum_{n\in\mathbb{P}}|g(x-u(n)a)|^2\,dx\\ &=\frac{1}{|b|}\int_K|f(x)|^2\,G(x)dx.\end{aligned}$$

Since this equality holds for all $f\in L^2(\Gamma)$, for any ball Γ of diameter at most $|b|^{-1}$, it follows that $G(x)=|b|$ for a.e. $x\in K$. Therefore,

$$\sum_{m,n\in\mathbb{P}}|<f,M_{u(m)b}T_{u(n)a}g>|^2=\frac{1}{|b|}\int_K|f(x)|^2\,G(x)dx,$$

for all functions $f\in L^2(K)$. By using Lemma 2.7 again, we have for all bounded, compactly supported $f\in L^2(K)$,

$$\frac{1}{|b|}\sum_{k\in\mathbb{P}/\{0\}}\int_K\overline{f(x)}f(x-u(k)b^{-1})\sum_{n\in\mathbb{P}}g(x-u(n)a)\overline{g(x-u(n)a-b^{-1}u(k))}\,dx=0. \quad (3.1)$$

changing the variable x by $x+u(k)b^{-1}$ and then replacing $u(k)$ by $-u(k)$ (which is allowed), we obtain

$$\frac{1}{|b|}\sum_{k\in\mathbb{P}/\{0\}}\int_K\overline{f(x-u(k)b^{-1})}f(x)\sum_{n\in\mathbb{P}}g(x-u(n)a-b^{-1}u(k))\overline{g(x-u(n)a)}\,dx=0. \quad (3.2)$$

Eq. (3.2) is complex conjugate of Eq. (3.1). Therefore adding (3.1) and (3.2), we get

$$\sum_{k\in\mathbb{P}/\{0\}}\mathrm{Re}\left(\int_K\overline{f(x)}f(x-u(k)b^{-1})\right.$$
$$\left.\times\sum_{n\in\mathbb{P}}g(x-u(n)a)\overline{g(x-u(n)a-b^{-1}u(k))}\,dx\right)=0. \quad (3.3)$$

Now fix $k_0 \geq 1$ and let Γ be any ball in K of diameter at most $\frac{1}{|b|}$. Define a function $f \in L^2(K)$ by

$f(x) = e^{-i \arg G_{k_o}(x)}$
and $f(x - b^{-1}u(k_0)) = 1$ for all $x \in \Gamma$ and $f(x) = 0$, otherwise. Then, by (3.3), we have

$$0 = \sum_{k \in \mathbb{P}\{0\}} \mathrm{Re}\Big(\int_K \overline{f(x)} f(x - u(k)b^{-1})$$
$$\times \sum_{n \in \mathbb{P}} g(x - u(n)a)\overline{g(x - u(n)a - b^{-1}u(k))}\, dx\Big)$$
$$= \mathrm{Re}\Big(\int_K \overline{f(x)} f(x - u(k_0)b^{-1}) G_{k_0}(x) dx\Big) = \int_\Gamma |G_{k_0}(x)|\, dx.$$

It follows that $G_{k_0}(x) = 0$, a.e. on Γ. Since Γ was an arbitrary ball of diameter at most $|b|^{-1}$, we conclude that $G_{k_o}(x) = 0$. A direct computation shows that
$G_{-k_0}(x) = \overline{G_{k_0}(x + b^{-1}u(k_0))} = 0,$
and we have proved statement (b) in (ii) for all $k \neq 0$.
(ii) $\Rightarrow$ (i): The assumption in (ii) imply, again by Lemma 2.7, that for all bounded, compactly supported $f \in L^2(K)$,

$$\sum_{m,n \in \mathbb{P}} |< f, M_{u(m)b} T_{u(n)a} g >|^2 = \frac{1}{|b|} \int_K |f(x)|^2 \sum_{n \in \mathbb{P}} |g(x - u(n)a)|^2\, dx = \| f \|^2 .$$

Since the bounded compactly supported functions are dense in $L^2(K)$, the conclusion follows by (2.4).

(ii) $\Rightarrow$ (iii): By Lemma 3.1.(ii), the statement (b) in (ii) is equivalent to $g \perp M_{u(m)a^{-1}} T_{u(n)b^{-1}} g$ for all $m \in \mathbb{P}, n \neq 0$. Using Lemma 3.1.(i) with $n = 0$, the function G is constant if and only if $g \perp M_{u(m)a^{-1}} g$ for all $m \neq 0$; and if this is the case, the relationship between $\| g \|^2$ and $G(x)$ follows from

$$\| g \|^2 = \int_K |g(x)|^2\, dx = \int_{G_\alpha} \sum_{n \in \mathbb{P}} |g(x - u(n)a)|^2\, dx$$
$$= \int_{G_\alpha} G(x) dx$$
$$= |\alpha| G(x)$$

(iii) $\Rightarrow$ (iv): This follows from the observation that for all $m,n,l,k \in \mathbb{P}$,

$$\left\langle M_{u(m)a^{-1}} T_{u(n)b^{-1}} g, M_{u(k)a^{-1}} T_{u(l)b^{-1}} g \right\rangle = \left\langle g, T_{-u(n)b^{-1}} M_{(u(k)-u(m))a^{-1}} T_{u(l)b^{-1}} g \right\rangle$$
$$= \left\langle g, \chi_{(u(k)-u(m))u(n)}(xa^{-1}b^{-1}) M_{(u(k)-u(m))a^{-1}} T_{(u(l)-u(n))b^{-1}} g \right\rangle$$
$$= \overline{\chi_{(u(k)-u(m))u(n)}} \left\langle g, (xa^{-1}b^{-1}) M_{(u(k)-u(m))a^{-1}} T_{(u(l)-u(n))b^{-1}} g \right\rangle.$$

For the final part of the theorem, we just observe that if $\left\{ M_{u(m)b} T_{u(n)a} g \right\}_{m,n\in\mathbb{P}}$ is a tight frame with frame bound 1, then for any $(m', n') \in \mathbb{P}^2$,

$$\left\| M_{u(m')b} T_{u(n')a} g \right\|^2 .$$
$$= \sum_{m,n\in\mathbb{P}} |< M_{u(m')b} T_{u(n')a} g, M_{u(m)b} T_{u(n)a} g >|^2$$
$$= \left\| M_{u(m')b} T_{u(n')a} g \right\|^4 + \sum_{(m,n)\neq(m',n')} |< M_{u(m')b} T_{u(n')a} g, M_{u(m)b} T_{u(n)a} g >|^2$$

If $||g|| = 1$ it follows from here that $\left\{ M_{u(m)b} T_{u(n)a} g \right\}_{m,n\in\mathbb{P}}$ is an orthonormal system.

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