DOI: 10.15415/mjis.2015.41006

The Interplay between l-max, l-min, p-max and p-min Stable Distributions

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Received: July 10, 2015| Revised: September 9, 2015| Accepted: September 17, 2015

Published online: September 30, 2015

Abstract: Extreme value laws are limit laws of linearly normalized partial maxima of independent and identically distributed (iid) random variables (rvs), also called as *l*-max stable laws. Similar to *l*-max stable laws, we have the *l*-min stable laws which are the limit laws of centered and scaled partial minima, *p*-max and *p*-min stable laws which are respectively the limit laws of normalized maxima and minima under power normalization. In this article, we look at transformations between *l*-max, *l*-min, *p*-max and *p*-min stable distributions and their domains. The transformations in this article are useful in simulation studies.

Mathematics Subject Classification: Primary 60G70, secondary 60E05.

Keywords and Phrases: *l*-max stable laws, *l*-min stable laws, *p*-max stable laws, *p*-min stable laws, domains of attraction.

1. INTRODUCTION

Extreme value theory is a classical topic in probability theory and mathematical statistics. The field of extremes has attracted the attention of engineers, scientists, actuaries and statisticians for many years. The fundamental result in extreme value theory is the form of limit distributions for centered and scaled maxima/minima. Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed (iid) random variables (rvs) with distribution function (df) F and $M_n = \max\{X_1, \ldots, X_n\}$. Suppose that there exists norming constants $a_n > 0$ and $b_n \in R, R$ the real line, such that $\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x), x \in C(G)$, the set of

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all continuity points of a nondegenerate df G. We then say that the df F belongs to the l-max domain of attraction of G and denote this by $F \in D_{l-\max}(G)$. The limit dfs G are the well known extreme value laws and G can be only one of three types of extreme value dfs, namely, (see, for example, [5]):

the Frèchet law, $\Phi_\alpha(x) = \exp\left\{-x^{-\alpha}\right\}, x \geq 0;$

the Weibull law, $\Psi_\alpha(x) = \exp\left\{-(-x)^{\alpha}\right\}, x < 0;$

the Gumbel law, $\Lambda(x) = \exp\left\{-\exp(-x)\right\}, x \in R,$

where $\alpha > 0$ is a parameter and dfs are given here and elsewhere in this article only for x values for which they belong to (0,1). The extreme value dfs G satisfy the stability property $G^n(a_n x + b_n) = G(x), x \in R,$ for constants $a_n > 0, b_n \in R, n \geq 1$ and were called l-max stable laws in [7], l standing for linear, meaning that normalization is linear. Here, two dfs F and G are said to be of the same type if $F(x) = G(Ax + B)$ for all x, for constants $A > 0$ and $B \in R$. We say that F belongs to the l-min domain of attraction of the nondegenerate df L under linear normalization and denote it by $F \in D_{l-\min}(L)$ if there exist norming constants $c_n > 0$ and $d_n \in R$ such that

$$\lim_{n\to\infty} P\left(\frac{m_n - d_n}{c_n} \leq x\right) = 1 - \lim_{n\to\infty}(1 - F(c_n x + d_n))^n = L(x), \quad x \in C(L)$$

where $m_n = \min\left\{X_1, \ldots, X_n\right\}$ The df L is called l-min stable df and can be only one of the following three types of dfs (see, for example, [5]):

negative Frèchet law, $L_{1,\alpha}(x) = 1 - \exp\left\{-(-x)^{-\alpha}\right\}, x < 0;$

negative Weibull law, $L_{2,\alpha}(x) = 1 - \exp\left\{-x^{\alpha}\right\}, 0 \leq x;$

negative Gumbel law, $L_3(x) = 1 - \exp\left\{-e^{x}\right\}, x \in R.$

There are several references for extreme value distributions under linear normalization. We name a few, [1, 3-6, 9]. Similar to l-max and l-min stable laws we have the p-max and p-min stable laws which are respectively the limit laws of normalized partial maxima and partial minima under power normalization. A nonlinear normalization called the power normalization was introduced in [8]. A df F is said to belong to the p-max domain of attraction of a nondegenerate df H under power normalization, denoted by $F \in D_{p-\max}(H)$ if there exist norming constants $\alpha_n > 0$ and $\beta_n > 0$ such that

$$\lim_{n\to\infty} P\left(\left|\frac{M_n}{\alpha_n}\right|^{\frac{1}{\beta_n}} \operatorname{sign}(M_n) \leq x\right) = \lim_{n\to\infty} F^n\left(\alpha_n \,|\, x\,|^{\beta_n} \operatorname{sign}(x)\right) = H(x), x \in C(H),$$

sign $(x) = -1, 0$ *or* 1

according as $x < 0, = 0$ *or* > 0 respectively. The df H is called *p*-max stable df (see, for example, [7]). It is known that the *p*-max stable dfs can be a *p*-type of only one of the six dfs, namely,

log-Frèchet law, $H_{1,\alpha}(x) = \exp\left\{-(\log x)^{-\alpha}\right\}, 1 \le x;$

log-Weibull law, $H_{2,\alpha}(x) = \exp\left\{-(-\log x)^{\alpha}\right\}, 0 \le x < 1;$

inverse log-Frèchet law, $H_{3,\alpha}(x) = \exp\left\{-(-\log(-x))^{-\alpha}\right\}, -1 \le x < 0;$

inverse log-Weibull law, $H_{4,\alpha}(x) = \exp\left\{-(\log(-x))^{-\alpha}\right\}, x < -1;$

standard Frèchet law, $\Phi(x) = \Phi_1(x), x \in R;$

standard Weibull law, $\Psi(x) = \Psi_1(x), x \in R,$

where $\alpha > 0$ is a parameter. Here, two dfs F and G are said to be of the same *p*-type if $F(x) = G(A\,|x|^B \text{ sign}(x))$ for all x, for constants $A > 0, B > 0$. We say that F belongs to the *p*-min domain of attraction of a nondegenerate df K under power normalization and denote it by $F \in D_{p-\min}(K)$ if there exist norming constants $\gamma_n > 0$ and $\delta_n > 0$ such that

$$\lim_{n\to\infty} P\left(\left|\frac{m_n}{\gamma_n}\right|^{\frac{1}{\delta_n}} sign(x) \le x\right) = 1 - \lim_{n\to\infty}\left\{1 - F(\gamma_n\,|\text{x}|^{\delta_n}\ \text{sign(x)})\right\}^n = K(\text{x}),\ \ \text{x} \in \text{C(K)}.$$

The *p*-min stable dfs can be *p*-types of the following six dfs:

negative log-Frèchet law, $K_{1,\alpha}(x) = 1 - \exp\left\{-\left(\log(-x)\right)^{-\alpha}\right\}, x < -1;$

negative log-Weibull law, $K_{2,\alpha}(x) = 1 - \exp\left\{-\left(-\log(-x)\right)^{\alpha}\right\}, -1 \le x < 0;$

inverse negative log-Frèchet law, $K_{3,\alpha}(x) = 1 - \exp\left\{-\left(-\log x\right)^{-\alpha}\right\}, 0 \le x < 1;$

inverse negative log-Weibull law, $K_{4,\alpha}(x) = 1 - \exp\left\{-\left(\log x\right)^{\alpha}\right\}, 1 \le x;$

standard negative Frèchet law, $K_5(x) = L_{1,1}(x), x \in R;$

standard exponential law, $K_6(x) = L_{2,1}(x), x \in R.$

In this article, we look at transformations between *l*-max, *l*-min, *p*-max and *p*-min stable distributions and their domains. Eighteen families of extremal stable laws are considered for study. The mapping that maps a rv within one family to a rv within another family is constructed for all pairs of families. And the transformations that map a max/min stable rv to a max/min stable rv of a different family are new. Section 3 contains the relationship among *l*-max, *l*-min, *p*-max and *p*-min stable distributions. In Section 4, examples for dfs in the domain of attraction of p-max are given. For easy understanding, the interrelations are tabulated in Tables 1 through 7. We denote $\max(a,b) = a \vee b$ for $a \in R, b \in R$.

2. INTERPLAY BETWEEN *l*-MAX, *l*-MIN, *p*-MAX AND *p*-MIN STABLE DISTRIBUTIONS

In this section the relationship among domains of attraction of *l*-max, *l*-min, *p*-max and *p*-min stable distributions are given as theorems and the results are tabulated for easy understanding in Table 1. Let rvs X and Y have respective dfs F and G and $a>0$ be a constant close to the right extremity of the corresponding df wherever applicable.

Theorem 2.1

(i) $X \sim F \in D_{l\text{-}\max}(\Phi_\alpha) \Rightarrow Y = e^X \sim G \in D_{p-\max}(H_{1,\alpha}), and$
$Y \sim G \in D_{p-\max}(H_{1,a}) \Rightarrow X = \log(a \vee Y) \sim F \in D_{l-\max}(\Phi_a).$

(ii) $X \sim F \in D_{l\text{-}\max}(\Psi_\alpha) \Rightarrow Y = e^X \sim G \in D_{p-\max}(H_{2,\alpha}), and$
$Y \sim G \in D_{p-\max}(H_{2,a}) \Rightarrow X = \log(a \vee Y) \sim F \in D_{l-\max}(\Psi_a).$

(iii) $X \sim F \in D_{l\text{-}\max}(\Lambda) \Rightarrow Y = e^X \sim G \in D_{p-\max}(\Phi), and$
$Y \sim G \in D_{p-\max}(\Phi) \Rightarrow X = \log(a \vee Y) \sim F \in D_{l-\max}(\Lambda).$

(iv) $X \sim F \in D_{l\text{-}\min}(L_{1,\alpha}) \Rightarrow Y = e^X \sim G \in D_{p-\min}(K_{3,\alpha}), and$
$Y \sim G \in D_{p-\min}(K_{3,a}) \Rightarrow X = \log(a \vee Y) \sim F \in D_{l-\min}(L_{1,a}).$

(v) $X \sim F \in D_{l\text{-}\min}(L_{2,\alpha}) \Rightarrow Y = e^X \sim G \in D_{p\text{-}\min}(K_{4,\alpha}), and$
$Y \sim G \in D_{p-\min}(K_{4,a}) \Rightarrow X = \log(a \vee Y) \sim F \in D_{l-\min}(L_{2,a}).$

(vi) $X \sim F \in D_{l\text{-}\min}(L_3) \Rightarrow Y = e^X \sim G \in D_{p-\min}(K_6), and$
$Y \sim G \in D_{p-\min}(K_6) \Rightarrow X = \log(a \vee Y) \sim F \in D_{l-\min}(L_3).$.

Remark: Statements (i), (ii) and (iii) in the above theorem can be proved as in [2]. We prove (iv) and proofs of (v) and (vi) follow on similar lines and are omitted.

Proof of (iv). Let $X \sim F \in D_{l-\min}(L_{1,\alpha})$ with norming constants $c_n > 0$ and $d_n \in R$. The df of $Y = e^X$ is given by $G(x) = P(Y \le x) = P(e^X \le x) = F(\log x), 0 \le x$. So,with $\gamma_n = e^{d_n}$ and $\delta_n = c_n$ $\left(1 - G\left(\gamma_n |x|^{\delta_n} \text{ sign(x)}\right)\right)^n = \left(1 - F(\log(\gamma_n x^{\delta_n})\right)^n = (1 - F(c_n \log(x)) + d_n)^n$, $x > 0$ Thus $1 - \lim_{n\to\infty}(1 - G\left(\gamma_n |x|^{\delta_n} \text{ sign(x)}\right)^n = L_{1,\alpha}(\log(x)) = K_{3,\alpha}(x)$, proving that $G \in D_{p-\min}(K_{3,\alpha})$.

Let $Y \sim G \in D_{p-\min}(K_{3,\alpha})$ with norming constants $\gamma_n > 0$ and $\delta_n > 0$. Then $X = \log(a \vee Y)$ has df $F(x) = P(X \le x) = P(\log(a \vee Y) \le x) = G(e^x), \log a < x$, so that with $c_n = \delta_n$ and $d_n = \log \gamma_n, (1 - F(c_n x + d_n))^n = (1 - G(e^{c_n x + d_n}))^n = (1 - G(\gamma_n (e^x)^{\delta_n}))^n$.

Thus $1-\lim_{n\to\infty}(1-F(c_n x+d_n))^n = K_{3,\alpha}(e^x) = L_{1,\alpha}(x)$, proving that $F\in D_{l-\min}(L_{1,\alpha})$ □

Theorem 2.2.

(i) $X\sim F\in D_{l\text{-}\max}(\Phi_\alpha)\Leftrightarrow Y=-e^X\sim G\in D_{p-\min}(K_{1,\alpha})$.

(ii) $X\sim F\in D_{l\text{-}\max}(\Psi_\alpha)\Leftrightarrow Y=-e^X\sim G\in D_{p-\min}(K_{2,\alpha})$.

(iii) $X\sim F\in D_{l\text{-}\max}(\Lambda)\Leftrightarrow Y=-e^X\sim G\in D_{p-\min}(K_5)$.

(iv) $X\sim F\in D_{l\text{-}\min}(L_{1,\alpha})\Leftrightarrow Y=-e^X\sim G\in D_{p\text{-}\max}(H_{3,\alpha})$.

(v) $X\sim F\in D_{l\text{-}\min}(L_{2,\alpha})\Leftrightarrow Y=-e^X\sim G\in D_{p\text{-}\max}(H_{4,\alpha})$.

(vi) $X\sim F\in D_{l\text{-}\min}(L_3)\Leftrightarrow Y=-e^X\sim G\in D_{p\text{-}\max}(\Psi)$.

Remark: We prove (i) and proofs of (ii) and (iii) follow on similar lines and are omitted. And (iv), (v) and (vi) can be proved as in [2] and the details are omitted.

Proof of (i): Let $X\sim F\in D_{l-\max}(\Phi_\alpha)$ with norming constants $a_n>0$ and $b_n\in R$. The df of $Y=-e^X$ is given by $G(x)=P(Y\le x)=P(-e^X\le x)=1-F(\log(-x)), x<0$. So, with $\gamma_n=e^{a_n}$ and $\delta_n=b_n$,

$$1-\lim_{n\to\infty}\left(1-G\left(\gamma_n|x|^{\delta_n}\ \mathrm{sign(x)}\right)\right)^n = F^n(\log(\gamma_n(-x)^{\delta_n}))$$

$=F^n(a_n\log(-x)+b_n), x<0$. Then $1-\lim_{n\to\infty}\left(1-G\left(\gamma_n|x|^{\delta_n}\ \mathrm{sign}(x)\right)\right)^n=$ $1-\Phi_\alpha(\log(-x))=K_{1,\alpha}(x), x<-1$, proving that $G\in D_{p-\min}(K_{1,\alpha})$.

Let $Y\sim G\in D_{p-\min}(K_{1,\alpha})$ with norming constants $\gamma_n>0$ and $\delta_n>0$. The df of $X=\log(-Y)$ is given by $F(x)=P(X\le x)=P(\log(-Y)\le x)=1-G(-e^x), x\in R$, so that with $a_n=\delta_n$ and $b_n=\log\gamma_n, F^n(a_n x+b_n)=\left(1-G(-e^{a_n x+b_n})\right)^n=\left(1-G(\gamma_n(-e^x)^{\delta_n})\right)^n$. Hence $\lim_{n\to\infty}F^n(a_n x+b_n)=1-K_{1,\alpha}(-e^x)=\Phi_\alpha(x), x>0$, proving that $F\in D_{l-\max}(\Phi_\alpha)$.

Theorem 2.3.

(i) $X \sim F \in D_{l-\max}(\Phi_\alpha) \Rightarrow Y = e^{-X} \sim G \in D_{p-\min}(K_{3.\alpha})$, and

$Y \sim G \in D_{p-\min}(K_{3,\alpha}) \Rightarrow X = -\log(a \vee Y) \sim F \in D_{l-\max}(\Phi_\alpha)$.

(ii) $X \sim F \in D_{l-\max}(\Psi_\alpha) \Rightarrow Y = e^{-X} \sim G \in D_{p-\min}(K_{4.\alpha})$, and

$Y \sim G \in D_{p-\min}(K_{4,\alpha}) \Rightarrow X = -\log(a \vee Y) \sim F \in D_{l-\max}(\Psi_\alpha)$.

(iii) $X \sim F \in D_{l-\max}(\Lambda) \Rightarrow Y = e^{-X} \sim G \in D_{p-\min}(K_6)$, and

$Y \sim G \in D_{p-\min}(K_6) \Rightarrow X = -\log(a \vee Y) \sim F \in D_{l-\max}(\Lambda)$.

(iv) $X \sim F \in D_{l-\min}(L_{1,\alpha}) \Rightarrow Y = e^{-X} \sim G \in D_{p-\max}(H_{1,\alpha})$, and

$Y \sim G \in D_{p-\max}(H_{1,\alpha}) \Rightarrow X = -\log(a \vee Y) \sim F \in D_{l-\min}(L_{1,\alpha})$.

(v) $X \sim F \in D_{l-\min}(L_{2,\alpha}) \Rightarrow Y = e^{-X} \sim G \in D_{p-\max}(H_{2,\alpha})$, and

$Y \sim G \in D_{p-\max}(H_{2,\alpha}) \Rightarrow X = -\log(a \vee Y) \sim F \in D_{l-\min}(L_{2,\alpha})$.

(vi) $X \sim F \in D_{l-\min}(L_3) \Rightarrow Y = e^{-X} \sim G \in D_{p-\max}(\Phi)$, and

$Y \sim G \in D_{p-\max}(\Phi) \Rightarrow X = -\log(a \vee Y) \sim F \in D_{l-\min}(L_3)$.

Remark: We prove (i) and (iv), the proofs of (ii) and (iii) follow on lines similar to the proof of (i) and proofs of (v) and (vi) follow on lines similar to the proof of (iv) and are omitted.

Proof of (i): Let $X \sim F \in D_{l-\max}(\Phi_\alpha)$ with norming constants $a_n > 0$ and $b_n \in R$. The df of $Y = e^{-X}$ is given by

$$G(x) = P(Y \le x) = P(e^{-X} \le x) = 1 - F(-\log x), 0 \le x, \tag{2.1}$$

so that with $\gamma_n = e^{-a_n}$ and $\delta_n = b_n$,

$\left(1 - G\left(\gamma_n x^{\delta_n}\right)\right)^n = \left(F\left(-\log\left(\gamma_n x^{\delta_n}\right)\right)\right)^n = F^n(a_n(-\log x) + b_n)), x > 0$. Then $1 - \lim_{n\to\infty}\left(1 - G(\gamma_n x^{\delta_n})\right)^n = 1 - \Phi_\alpha(-\log x) = K_{3,\alpha}(x), 0 < x < 1$, proving that $G \in D_{p-\min}(K_{3,\alpha})$

Let $Y \sim G \in D_{p-\min}(K_{3,\alpha})$. The df of $X = -\log(a \vee Y)$ is given by

$$F(x) = P(X \le x) = P(-\log(a \vee Y) \le x) = 1 - G(e^{-x}), x \le -\log a, \tag{2.2}$$

so that with $a_n = -\log \gamma_n, b_n = \delta_n$,

$$F^n(a_n x + b_n) = (1 - G(e^{-(a_n x + b_n)})^n = \left(1 - G\left(\gamma_n (e^{-x})^{\delta_n}\right)\right)^n, x < \log\left(\frac{\gamma_n}{a}\right)^{1/\delta_n}.$$

Then $\lim_{n\to\infty} F^n(a_n x + b_n) = 1 - K_{3,\alpha}(e^{-x}) = \Phi_\alpha(x), x > 0$, proving that $F \in D_{l-\max}(\Phi_\alpha)$

Proof of (iv): Let $X \sim F \in D_{l-\min}(L_{1,\alpha})$ with norming constants $c_n > 0$ and $d_n \in R$. Then from (2.1), with $\delta_n = c_n$ and $\gamma_n = e^{-d_n}, G^n\left(\gamma_n x^{\delta_n}\right) = \left(1 - F\left(-\log\left(\gamma_n x^{\delta_n}\right)\right)\right)^n = \left(1 - F\left(c_n\left(-\log x\right) + d_n\right)\right)^n, x > 0.$
So $\lim_{n\to\infty} G(\gamma_n x^{\delta_n}) = 1 - L_{1,\alpha}(-\log x) = H_{1,\alpha}(x), x > 1$, proving that $F \in D_{l-\max}(H_{1,\alpha})$.

Let $Y \sim G \in D_{p-\max}(H_{1,\alpha})$ with norming constants $\alpha_n > 0$ and $\beta_n > 0$. Then from (2.2), with

$$\alpha_n = e^{-d_n}, \beta_n = c_n, 1 - \left(1 - F\left(c_n x + d_n\right)\right)^n = 1 - G^n\left(e^{-(c_n x + d_n)}\right)$$

$$) = 1 - G^n\left(\alpha_n (e^{-x})^{\beta_n}\right), x \le \log\left(\frac{\alpha_n}{a}\right)^{\frac{1}{\beta_n}}.$$ So $1 - \lim_{n\to\infty}\left(1 - F(c_n x + d_n)\right)^n$

$= 1 - H_{1,\alpha}(e^{-x}) = L_{1,\alpha}(x), x < 0$, proving (iv). □

Theorem 2.4.

(i) $X \sim F \in D_{l-\max}(\Phi_\alpha) \Leftrightarrow Y = -e^{-X} \sim G \in D_{p-\max}(H_{3,\alpha})$.

(ii) $X \sim F \in D_{l-\max}(\Psi_\alpha) \Leftrightarrow Y = -e^{-X} \sim G \in D_{p-\max}(H_{4,\alpha})$.

(iii) $X \sim F \in D_{l-\max}(\Lambda) \Leftrightarrow Y = -e^{-X} \sim G \in D_{p-\max}(\Psi)$.

(iv) $X \sim F \in D_{l-\min}(L_{1,\alpha}) \Leftrightarrow Y = -e^{-X} \sim G \in D_{p-\min}(K_{1,\alpha})$.

(v) $X \sim F \in D_{l-\min}(L_{2,\alpha}) \Leftrightarrow Y = -e^{-X} \sim G \in D_{p-\min}(K_{2,\alpha})$.

(vi) $X \sim F \in D_{l-\min}(L_3) \Leftrightarrow Y = -e^{-X} \sim G \in D_{p-\min}(K_S)$.

Remark: We prove (i) and (iv), the proofs of (ii) and (iii) follow on lines similar to the proof of (i) and the proofs of (v) and (vi) follow on lines similar to the proof of (iv) and are omitted.

Proof of (i). Let $X \sim F \in D_{l-\max}(\Phi_\alpha)$ with norming constants $a_n > 0$ and $b_n \in R$. The df of $Y = -e^{-X}$ is given by

$$G(x) = P(Y \le x) = P(-e^{-X} \le x) = F(-\log(-x)), x < 0, \qquad (2.3)$$

so that with $\beta_n = a_n, \alpha_n = e^{-b_n}$,

$G^n\left(\alpha_n \,|\, x\,|^{\beta_n} \text{ sign(x)}\right) = F^n(-\log(\alpha_n(-x)^{\beta_n})) = F^n\left(a_n(-\log(-x)\right) + b_n), x < 0.$

So $\lim_{n\to\infty} G^n(\alpha_n \,|\, x\,|^{\beta_n} \text{ sign}(x)) = \Phi_\alpha(-\log(x)) = H_{3,\alpha}(x), -1 < x < 0$, proving that $G \in D_{p-\max}(H_{3,\alpha})$.

Let $Y \sim G \in D_{p-\max}(H_{3,\alpha})$ with norming constants $\alpha_n > 0$ and $\beta_n > 0$. The df of $X = -\log(-Y)$ is given by

$$F(x) = P(X \le x) = P(-\log(-Y) \le x) = G(-e^{-x}), x \in R, \qquad (2.4)$$

so that with $a_n = \beta_n$ and $b_n = -\log \alpha_n$,

$F^n(a_n x + b_n) = G^n(-e^{-(a_n x + b_n)}) = G^n(\alpha_n(-e^{-x})^{\beta_n})$ and

$\lim_{n\to\infty} F^n(a_n x + b_n) = H_{3,\alpha}(-e^{-x}) = \Phi_\alpha(x), x > 0$, proving (i).

Proof of (iv): Let $X \sim F \in D_{l-\min}(L_{1\alpha})$ with norming constants $c_n > 0$ and $d_n \in R$. Then from (2.3), $\left(1 - G\left(\gamma_n \,|\, x\,|^{\delta_n} \; sign(x)\right)\right)^n = \left(1 - F\left(-\log(\gamma_n(-x)^{\delta_n})\right)\right)^n =$ $\left(1 - F(c_n(-\log(-x)\right) + d_n))^n, x < 0$, so that with $\gamma_n = e^{-d_n}, \delta_n = c_n$,

$1 - \lim_{n\to\infty}\left(1 - G(\gamma_n \,|\, x\,|^{\delta_n} \text{ sign}(x)\right)^n = L_{1,\alpha}(-\log(-x)) = K_{1,\alpha}(x), x < -1,$

proving that $G \in D_{p-\min}(K_{1,\alpha})$.

Let $Y \sim G \in D_{p-\min}(K_{1,\alpha})$ with norming constants $\gamma_n > 0$ and $\delta_n > 0$. Then from (2.4), $(1 - F(c_n x + d_n))^n = (1 - G(-e^{-(c_n x + d_n)}))^n = \left(1 - G\left(\gamma_n x^{\delta_n}\right)\right)^n$, so that with $c_n = \delta_n$ and

$d_n = -\log \gamma_n, 1 - \lim_{n\to\delta}(1 - F(c_n x + d_n))^n = K_{1,\alpha}(-e^{-x}) = L_{1,\alpha}(x)$, proving (iv). □

The following table summarizes the relationship between domains of attraction of l-max, l-min p-max and p-min stable distributions.

Table 1: Relationship between domains of attraction of l-max, l-min, p-max and p-min stable distributions.

$X \in \downarrow \Rightarrow$	$e^X \in$	$e^{-X} \in$	$-e^X \in$	$-e^{-X} \in$
$D_{l-\max}(\Phi_\alpha)$	$D_{p-\max}(H_{1,\alpha})$	$D_{p-\min}(K_{3,\alpha})$	$D_{p-\min}(K_{1,\alpha})$	$D_{p-\max}(H_{3,\alpha})$
$D_{l-\max}(\Psi_\alpha)$	$D_{p-\max}(H_{2,\alpha})$	$D_{p-\min}(K_{4,\alpha})$	$D_{p-\min}(K_{2,\alpha})$	$D_{p-\max}(H_{4,\alpha})$
$D_{l-\max}(\Lambda)$	$D_{p-\max}(\Phi)$	$D_{p-\min}(K_6)$	$D_{p-\min}(K_5)$	$D_{p-\max}(\Psi)$
$D_{l-\min}(L_{1,\alpha})$	$D_{p\text{-}min}(K_{3,\alpha})$	$D_{p\text{-}max}(H_{1,\alpha})$	$D_{p\text{-}max}(H_{3,\alpha})$	$D_{p\text{-}min}(K_{1,\alpha})$
$D_{l\text{-}min}(L_{2,\alpha})$	$D_{p\text{-}min}(K_{4,\alpha})$	$D_{p\text{-}max}(H_{2,\alpha})$	$D_{p\text{-}max}(H_{4,\alpha})$	$D_{p\text{-}min}(K_{2,\alpha})$
$D_{l\text{-}min}(L_3)$	$D_{p\text{-}min}(K_6)$	$D_{p\text{-}max}(\Phi)$	$D_{p\text{-}max}(\Psi)$	$D_{p\text{-}min}(K_5)$

The table is read as follows: The entry, say, in row 2, is read as : If $X \in D_{l-\max}(\Phi_\alpha)$ then $e^X \in D_{p-\max}(H_{1,\alpha}), e^{-x} \in D_{p-\min}(K_{3,\alpha}), -e^x \in D_{p-\min}(K_{1,\alpha})$ and $-e^{-X} \in D_{p-\max}(H_{3,\alpha})$.

3. TRANSFORMATIONS

The following tables give the interrelationship between l-max, l-min, p-max and p-min stable distributions. The tables may be read as follows: for example, in Table 2 below, the entry, say, in row 3 and column 2, is read as: If $X \sim \Psi_\alpha$ then $Y = -\frac{1}{X} \sim \Phi_\alpha$, and so on.

Table 2. Relationship between l-max and l-min stable distributions.

	$\Phi\alpha$	$\Psi\alpha$	Λ	$L_{1,\alpha}$	$L_{2,\alpha}$	L_3
Φ_α	X	$-\frac{1}{X}$	$\alpha \log X$	$-X$	$\frac{1}{X}$	$-\alpha \log X$
Ψ_α	$-\frac{1}{X}$	X	$-\alpha \log(-X)$	$\frac{1}{X}$	$-X$	$\alpha \log(-X)$
Λ	$\exp\left(\frac{X}{\alpha}\right)$	$-\exp\left(\frac{X}{\alpha}\right)$	X	$-\exp\left(\frac{X}{\alpha}\right)$	$\exp\left(\frac{-X}{\alpha}\right)$	$-X$
$L_{1,\alpha}$	$-X$	$\frac{1}{X}$	$\alpha \log(-X)$	X	$-\frac{1}{X}$	$-\alpha \log(-X)$

$L_{2,\alpha}$	$\frac{1}{X}$	$-X$	$-\alpha \log X$	$-\frac{1}{X}$	X	$\alpha \log X$
L_3	$\exp\left(-\frac{X}{\alpha}\right)$	$-\exp\left(\frac{X}{\alpha}\right)$	$-X$	$-\exp\left(-\frac{X}{\alpha}\right)$	$\exp\left(\frac{X}{\alpha}\right)$	X

Table 3. Relationship among *p*-max stable distributions.

	$H_{1,\alpha}$	$H_{2,\alpha}$	$H_{3,\alpha}$	$H_{4,\alpha}$	Φ	Ψ
$H_{1,\alpha}$	X	$e^{(-\log X)^{-1}}$	$-\frac{1}{X}$	$-e^{(\log X)^{-1}}$	$(\log X)^{\alpha}$	$-(\log X)^{-\alpha}$
$H_{2,\alpha}$	$e^{-(\log X)^{-1}}$	X	$-e^{(\log X)^{-1}}$	$-\frac{1}{X}$	$-(\log X)^{-\alpha}$	$-(-\log X)^{\alpha}$
$H_{3,\alpha}$	$-\frac{1}{X}$	$e^{(\log(-X))^{-1}}$	X	$-e^{-(\log(-X))^{-1}}$	$(-\log(-X))^{\alpha}$	$-(-\log(-X))^{-\alpha}$
$H_{4,\alpha}$	$e^{(\log(-X))^{-1}}$	$-\frac{1}{X}$	$-e^{\left(\log(\frac{1}{X})\right)^{-1}}$	X	$(\log(-X))^{-\alpha}$	$-(\log(-X))^{\alpha}$
Φ	$e^{X^{1/\alpha}}$	$e^{-\left(\frac{1}{X}\right)^{1/\alpha}}$	$-e^{-X^{1/\alpha}}$	$-e^{\left(\frac{1}{X}\right)^{1/\alpha}}$	X	$-\frac{1}{X}$
Ψ	$e^{\left(-\frac{1}{X}\right)^{1/\alpha}}$	$e^{-(-X)^{1/\alpha}}$	$-e^{-\left(-\frac{1}{X}\right)^{1/\alpha}}$	$-e^{(-X)^{1/\alpha}}$	$-\frac{1}{X}$	X

Table 4. Relationship between *l*-max/*l*-min and *p*-max stable distributions.

	$H_{1,\alpha}$	$H_{2,\alpha}$	$H_{3,\alpha}$	$H_{4,\alpha}$	Φ	Ψ
Φ_{α}	e^{X}	$e^{-\frac{1}{X}}$	$-e^{-X}$	$-e^{\frac{1}{X}}$	X^{α}	$-X^{-\alpha}$
Ψ_{α}	$e^{-\frac{1}{X}}$	e^{X}	$-e^{\frac{1}{X}}$	$-e^{-X}$	$(-X)^{-\alpha}$	$-(-X)^{\alpha}$
Λ	$\exp\left((e^{-X})^{-\frac{1}{\alpha}}\right)$	$\exp\left(-(e^{-X})^{\frac{1}{\alpha}}\right)$	$-\exp\left(-(e^{-X})^{-\frac{1}{\alpha}}\right)$	$-\exp\left(-(e^{-X})^{\frac{1}{\alpha}}\right)$	e^{X}	$-e^{-X}$
$L_{1,\alpha}$	e^{-X}	$e^{\frac{1}{X}}$	$-e^{X}$	$-e^{-\frac{1}{X}}$	$(-X)^{\alpha}$	$-(-X)^{-\alpha}$

$L_{2,\alpha}$	$e^{\frac{1}{X}}$	e^{-X}	$-e^{-\frac{1}{X}}$	$-e^{X}$	$X^{-\alpha}$	$-X^{\alpha}$
L_3	$\exp\left((e^{X})^{-\frac{1}{\alpha}}\right)$	$\exp\left(-(e^{X})^{\frac{1}{\alpha}}\right)$	$\exp\left(-(e^{X})^{-\frac{1}{\alpha}}\right)$	$-\exp\left((e^{X})^{\frac{1}{\alpha}}\right)$	e^{-X}	$-e^{X}$

Table 5. Relationship among *p*-min stable distributions.

	$K_{1,\alpha}$	$K_{2,\alpha}$	$K_{3,\alpha}$	$K_{4,\alpha}$	K_5	K_6
$K_{1,\alpha}$	X	$-e^{(-\log(-X))^{-1}}$	$-\frac{1}{X}$	$e^{(\log(-X))^{-1}}$	$-(\log(-X))^{\alpha}$	$(\log(-X))^{-\alpha}$
$K_{2,\alpha}$	$-e^{-(\log(-X))^{-1}}$	X	$e^{(\log(-X))^{-1}}$	$-\frac{1}{X}$	$-(-\log(-X))^{-\alpha}$	$(-\log(-X))^{\alpha}$
$K_{3,\alpha}$	$-\frac{1}{X}$	$-e^{(\log X)^{-1}}$	X	$e^{-(\log X)^{-1}}$	$-(-\log X)^{-\alpha}$	$(-\log X)^{-\alpha}$
$K_{4,\alpha}$	$-e^{(\log X)^{-1}}$	$-\frac{1}{X}$	$e^{\left(\log\left(\frac{1}{X}\right)\right)^{-1}}$	X	$(-\log X)^{-\alpha}$	$(\log X)^{\alpha}$
K_5	$-e^{(-X)^{1/\alpha}}$	$-e^{-\left(-\frac{1}{X}\right)^{1/\alpha}}$	$e^{-(-X)^{\frac{1}{\alpha}}}$	$e^{\left(-\frac{1}{X}\right)^{1/\alpha}}$	X	$-\frac{1}{X}$
K_6	$-e^{\left(\frac{1}{X}\right)^{1/\alpha}}$	$-e^{(-X)^{1/\alpha}}$	$e^{-\left(\frac{1}{X}\right)^{1/\alpha}}$	$e^{X^{1/\alpha}}$	$-\frac{1}{X}$	X

Table 6. Relationship between *l*-max/*l*-min and *p*-min stable distributions.

	$K_{1,\alpha}$	$K_{2,\alpha}$	$K_{3,\alpha}$	$K_{4,\alpha}$	K_5	K_6
Φ_α	$-e^{X}$	$-e^{-\frac{1}{X}}$	e^{-X}	$e^{\frac{1}{X}}$	$-X^{\alpha}$	$X^{-\alpha}$
Ψ_α	$-e^{-\frac{1}{X}}$	$-e^{X}$	$e^{\frac{1}{X}}$	e^{-X}	$-(-X)^{-\alpha}$	$(-X)^{\alpha}$
Λ	$-\exp\left((e^{-X})^{-\frac{1}{\alpha}}\right)$	$\exp\left(-(e^{-X})^{\frac{1}{\alpha}}\right)$	$\exp\left(-(e^{-X})^{-\frac{1}{\alpha}}\right)$	$\exp\left((e^{-X})^{-\frac{1}{\alpha}}\right)$	$-e^{X}$	e^{-x}
$L_{1,\alpha}$	$-e^{-X}$	$-e^{\frac{1}{X}}$	e^{X}	$e^{-\frac{1}{X}}$	$-(-X)^{\alpha}$	$(-X)^{-\alpha}$

$L_{2,\alpha}$	$-e^{\frac{1}{X}}$	$-e^{-X}$	$e^{-\frac{1}{X}}$	e^{X}	$-X^{-\alpha}$	X^{α}
L_3	$-\exp\left((e^{X})^{-\frac{1}{\alpha}}\right)$	$-\exp\left(-(e^{X})^{\frac{1}{\alpha}}\right)$	$\exp\left(-(e^{X})^{-\frac{1}{\alpha}}\right)$	$\exp\left((e^{X})^{\frac{1}{\alpha}}\right)$	$-e^{-X}$	e^{X}

Table 7. Relationship between p-max and p-min stable distributions.

	$K_{1,\alpha}$	$K_{2,\alpha}$	$K_{3,\alpha}$	$K_{4,\alpha}$	K_5	K_6
$H_{1,\alpha}$	$-X$	$-e^{-(\log X)^{-1}}$	$\frac{1}{X}$	$e^{(\log X)^{-1}}$	$-(\log X)^{\alpha}$	$(\log X)^{-\alpha}$
$H_{2,\alpha}$	$-e^{-(\log X)^{-1}}$	$-X$	$e^{(\log X)^{-1}}$	$\frac{1}{X}$	$-(-\log X)^{-\alpha}$	$(-\log X)^{\alpha}$
$H_{3,\alpha}$	$\frac{1}{X}$	$-e^{(\log(-X))^{-1}}$	$-X$	$e^{-(\log(-X))^{-1}}$	$-(-\log(-X))^{\alpha}$	$(-\log(-X))^{-\alpha}$
$H_{4,\alpha}$	$-e^{(\log(-X))^{-1}}$	$\frac{1}{X}$	$e^{\left(\log\left(-\frac{1}{X}\right)\right)^{-1}}$	$-X$	$-(\log(-X))^{-\alpha}$	$(\log(-X))^{\alpha}$
Φ	$-e^{X^{\frac{1}{\alpha}}}$	$-e^{-\left(\frac{1}{X}\right)^{\frac{1}{\alpha}}}$	$e^{-X^{1/\alpha}}$	$e^{-\left(-\frac{1}{X}\right)^{1/\alpha}}$	$-X$	$\frac{1}{X}$
Ψ	$-e^{\left(-\frac{1}{X}\right)^{\frac{1}{\alpha}}}$	$-e^{-(-X)^{\frac{1}{\alpha}}}$	$e^{-\left(-\frac{1}{X}\right)^{1/\alpha}}$	$e^{(-X)^{1/\alpha}}$	$\frac{1}{X}$	$-X$

4. SOME EXAMPLES FOR *p*-MAX AND *p*-MIN DOMAINS.

This section provides examples for dfs belonging to domains of attraction of *p*-max stable laws and *p*-min stable laws. Some examples of dfs in domain of attraction of *l*-max stable laws along with norming constants are given in [4]. One can get examples for *l*-min by using the result: if $F \in D_{l-\max}(G)$ with norming constants a_n and b_n then $F \in D_{l-\min}(L)$ with norming constants $c_n = a_n$ and $d_n = -b_n$ and with $L(x) = 1 - G(-x)$.

Examples for p-max domains: Pdfs are given with norming constants and value of the parameter of the limit law, wherever applicable.

1. Dfs in $D_{p-\max}(H_{1,\alpha})$:

a. Log-Cauchy with pdf $f(x)=\frac{1}{\pi x(1+(\log x)^2)}, 0<x<\infty, \alpha_n=1, \beta_n=\frac{n}{\pi}$ with $\alpha=1$.

b. Log-Pareto with pdf $f(x)=\frac{\beta}{x(\log x)^{\beta+1}}, x>e, \alpha_n=1, \beta_n=n^{\frac{1}{\beta}}$ with $\alpha=\beta$.

2. Dfs in $D_{p-\max}(H_{2,\alpha})$:

a. Uniform with pdf $f(x)=1, 0<x<1, \alpha_n=1, \beta_n=\frac{1}{n}$ with $\alpha=1$.

b. Beta with pdf

$$f(x)=\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}, 0<x<1, a,b>0, \alpha_n=1, \beta_n=\left(\frac{n}{b}\frac{1}{B(a,b)}\right)^{\frac{-1}{b}}$$

with $\alpha=b$.

c. Log-beta with pdf

$$f(x)=\frac{1}{B(a,b)x}(\log x)^{a-1}(1-\log x)^{b-1}, 0<x<e, a,b>0, \alpha_n$$

$$=e, \beta_n=\left(\frac{n}{b}\frac{1}{B(a,b)}\right)^{-\frac{1}{b}}$$

with $\alpha=b$.

3. Dfs in $D_{p-\max}(H_{3,\alpha})$

a. Inverse log-Cauchy with pdf

$$f(x)=\frac{-1}{\pi x\left(1+\left(\log\left(-\frac{1}{x}\right)\right)^2\right)}, x<0, \alpha_n=1, \beta_n=\frac{n}{\pi}$$

with $\alpha=1$

b. Inverse log-Pareto with pdf

$$f(x)=\frac{-\beta}{x\left(\log\left(-\frac{1}{x}\right)\right)^{\beta+1}}, -e^{-1}<x<0, \alpha_n=1, \beta_n=n^{\frac{1}{\beta}}$$

with $\alpha=\beta$

4. Dfs in $D_{p-\max}(H_{4,\alpha})$:

a. Inverse log-beta with pdf

$$f(x)=\frac{1}{B(a,b)x}\left(\log\left(-\frac{1}{x}\right)\right)^{a-1}\left(1-\log\left(-\frac{1}{x}\right)\right)^{b-1},x<-e^{-1}a,b>0,\alpha_n$$

$$=\frac{1}{\varepsilon},\beta_n=\left(\frac{n}{b}\frac{1}{B(a,b)}\right)^{-\frac{1}{b}} \text{ with } \alpha=b .$$

5. Dfs in $D_{p-\max}(\Phi)$:

a. Cauchy with pdf $f(x)=\frac{1}{\pi(1+x^2)},x\in R,\alpha_n=1,\beta_n=\frac{n}{\pi}$.

b. Normal with pdf $f(x)=\frac{1}{2\pi}e^{-\frac{x^2}{2},x\in R}$,

$$\alpha_n-\sqrt{2\log n}-\frac{\{\log 4\pi+\log\log n\}}{\sqrt{2\log n}},\beta_n=2\log n-\frac{\{\log 4\pi+\log\log n\}}{2}.$$

c. Gamma with pdf

$$f(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x},x>0,\alpha,\beta>0,\alpha_n=\frac{1}{\beta}(\log n+(\alpha-1))\log$$

$$\log n-\log\Gamma(\alpha)),\beta_n=(\log n+(\alpha-1))\log\log n-\log\Gamma(\alpha).$$

d. Log-gamma with pdf $f(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}(\log x)^{\alpha-1}x^{-\beta-1},x>1,\alpha,\beta>0,$

$$\alpha_n=\left(\frac{1}{\Gamma(\alpha)}(\log n)^{\alpha-1_n}\right)^{\frac{1}{\alpha}},\beta_n=\frac{1}{\alpha_n}\cdot$$

6. Dfs in $D_{p-\max}(\Psi)$:

a. Inverse log-gamma with pdf $f(x)=\frac{\alpha^\beta}{\Gamma(\alpha)x^2}\left(\log\left(-\frac{1}{x}\right)\right)^{\beta-1}\left(-\frac{1}{x}\right)^{-\alpha-1}$,

$$-1<x<0,\alpha,\beta>0,\alpha_n=\left(\frac{1}{\Gamma(\alpha)}(\log n)^{\alpha-1_n}\right)^{-\frac{1}{\alpha}},\beta_n=\frac{1}{\alpha_n}$$

b. Inverse gamma with pdf $f(x)=\frac{\alpha^\beta}{\Gamma(\alpha)x^2}\left(-\frac{1}{x}\right)^{\beta-1}e^{\frac{\alpha}{x}},x<0$,

$$\alpha_n = \left(\frac{1}{\beta}(\log n + (\alpha - 1)) \log\log n - \log\Gamma(\alpha) \right)^{-1}$$

$$\beta_n = \left(\log n + (\alpha - 1) \right) \log\log n - \log\Gamma(\alpha).$$

Remark: We know that if $F \in D_{p-\max}(H)$ with norming constants $\alpha_n > 0$ and $\beta_n > 0$ then $F \in D_{p-\min}(K)$ with norming constants $\gamma_n = \alpha_n$ and $\delta_n = \beta_n$ and with $K(x) = 1 - H(-x)$. In particular, $K_i(x) = 1 - H_i(-x), i = 1,2,3,4$ and $K_5(x) = 1 - \Phi(-x), K_6(x) = 1 - \Psi(-x)$. One can get examples for p-min domain by using this result.

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